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Best constants in second-order Sobolev inequalities on Riemannian manifolds and applications

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Abstract

Let (M,g) be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$ and $1 < p < n/2$. Considering the norm

$$\|u\| = \left(\|\Delta_g u\|^p_{L^p(M)} + \|u\|^p_{L^p(M)}\right)^{1/p}$$

on each of the spaces $H^{2,p}(M)$, $H_0^{2,p}(M)$ and $H^{2,p}(M) \cap H_0^{1,p}(M)$, we study an asymptotically sharp inequality associated to the critical Sobolev embedding of these spaces. As an application, we investigate the influence of the geometry in the existence of solutions for some fourth-order problems involving critical exponents on manifolds. In particular, new phenomena arise in Brezis–Nirenberg type problems on manifolds with positive scalar curvature somewhere, in contrast with the Euclidean case. We also show that on such manifolds the corresponding optimal inequality for $p = 2$ is not valid.

Résumé

Soient (M,g) une variété riemannienne compacte, à bord ou sans bord, de dimension $n \geqslant 3$ et $1 < p < n/2$. Considerant la norme

$$\|u\| = \left(\|\Delta_g u\|^p_{L^p(M)} + \|u\|^p_{L^p(M)}\right)^{1/p}$$

nous étudions une inégalité asymptotiquement précise associée à l'inclusion de Sobolev critique sur chacun des espaces $H^{2,p}(M)$, $H_0^{2,p}(M)$ et $H^{2,p}(M) \cap H_0^{1,p}(M)$. Comme application, nous examinons l'influence de la géométrie sur l'existence de solutions de quelques problèmes du

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quatrième ordre avec exposants critiques sur les variétés. En particulier, de nouveaux phénomènes surgissent dans les problèmes du type Brezis–Nirenberg sur les variétés dont certaines parties sont à courbure scalaire positive en contraste avec le cas euclidien. Nous montrons aussi que sur de telles variétés l'inégalité optimale correspondant à $p = 2$ n'est pas valable.

Keywords: Best constants; Sobolev inequalities; Compact Riemannian manifolds; Fourth-order elliptic equations; Critical Sobolev exponent; Concentration-compactness principle; Regularity

Mots-clés : Meilleures constantes ; Inégalités de Sobolev ; Variétés riemanniennes compactes ; Equations elliptiques du quatrième ordre ; Exposant critique de Sobolev

1. Introduction and main results

Best constants and sharp Poincaré–Sobolev type inequalities of first order on Riemannian manifolds, with or without boundary, have been extensively studied and considerable advance has been made in their understanding (see [3,20] for a list of references, and [4,15,21,24] for some recent results**)**. Although some open problems still remain, the next step forward has already been taken and questions related to second-order Sobolev inequalities have started to be investigated very recently, particularly in connection with Paneitz–Branson type operators, which were introduced in [8,29]. We mention the works [1,2,12,17], among others.

Let (M, g) be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$. For $1 < p < n/2$, we denote by $H_0^{1,p}(M)$ the standard first-order Sobolev space defined as the completion of $C_0^\infty(M)$ with respect to the norm

$$\|u\|_{H^{1,p}(M)} = \left(\int_M |\nabla_g u|^p \, \mathrm{d}v_g + \int_M |u|^p \, \mathrm{d}v_g\right)^{1/p},$$

and by $H_0^{2,p}(M)$ and $H^{2,p}(M)$ the standard second-order Sobolev spaces defined as the completion, respectively, of $C_0^\infty(M)$ and $C^\infty(M)$ with respect to the norm

$$\|u\|_{H^{2,p}(M)} = \left(\int_M \left|\nabla_g^2 u\right|^p \mathrm{d}v_g + \int_M |\nabla_g u|^p \, \mathrm{d}v_g + \int_M |u|^p \, \mathrm{d}v_g\right)^{1/p}.$$

In this work we consider the following Sobolev spaces:

$$E_1 = H^{2,p}(M),$$

if M has no boundary, and

$$E_2 = H_0^{2,p}(M), \qquad E_3 = H^{2,p}(M) \cap H_0^{1,p}(M),$$

if M has boundary. Denoting by $\Delta_g u = \mathrm{div}_g(\nabla u)$ the Laplacian with respect to the metric g, a norm on E_i equivalent to $\|\cdot\|_{H^{2,p}(M)}$ is:

$$\|u\|_{E_i} = \left(\int\limits_M |\Delta_g u|^p \,\mathrm{d}v_g + \int\limits_M |u|^p \,\mathrm{d}v_g\right)^{1/p}.$$

(For the convenience of the reader, a proof of this fact is included in Appendix A.) The Sobolev embedding theorem ensures that the inclusion $E_i \subset L^{p^*}(M)$ is continuous for $p^* = np/(n-2p)$. Thus, there exist constants $A, B \in \mathbb{R}$ such that

$$\|u\|^p_{L^{p^*}(M)} \leqslant A\|\Delta_g u\|^p_{L^p(M)} + B\|u\|^p_{L^p(M)} \tag{1}$$

for all $u \in E_i$. Consider, for each i, the first and second best constants associated to this inequality:

$$\mathcal{A}^i_p(M) = \inf\{A \in \mathbb{R}\colon \text{ there exists } B \in \mathbb{R} \text{ such that inequality (1) holds}\}$$

and

$$\mathcal{B}^i_p(M) = \inf\{B \in \mathbb{R}\colon \text{ there exists } A \in \mathbb{R} \text{ such that inequality (1) holds}\},$$

respectively. Two natural questions in this context are the dependence or not of the best constants on the geometry of the manifold M, and the validity or not of the associated optimal inequalities:

$$\|u\|^p_{L^{p^*}(M)} \leqslant \mathcal{A}^i_p(M)\|\Delta_g u\|^p_{L^p(M)} + B\|u\|^p_{L^p(M)} \tag{2}$$

and

$$\|u\|^p_{L^{p^*}(M)} \leqslant A\|\Delta_g u\|^p_{L^p(M)} + \mathcal{B}^i_p(M)\|u\|^p_{L^p(M)} \tag{3}$$

for all $u \in E_i$.

Concerning the second best constant and optimal inequality (3), work done by Bakry [5] and by Druet and Hebey (presented in [20]) on first-order Sobolev inequalities immediately generalizes to the second-order case, and one finds that

$$\mathcal{B}^i_p(M) = \mathrm{vol}_g(M)^{-p/(2n)}$$

and that (3) is valid if and only if $n = 3, 4$ or if $n \geqslant 5$ and $1 < p \leqslant 2$.

Similarly to what happens in the first-order case, the study of the first best constant $\mathcal{A}^i_p(M)$ and the optimal inequality (2) is more delicate. Recently, Djadli et al. [12] established, for $p = 2$ and M without boundary of dimension $n \geqslant 5$, the independence of the first best constant with respect to the geometry (see also Caraffa [10]). We show that $\mathcal{A}^i_p(M)$ is independent of the metric for $1 < p < n/2$ on any compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$. In order to state our results precisely, let us fix some notations. Let $D^{2,p}(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ under the norm

$$\|u\|_{D^{2,p}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\Delta u|^p \,\mathrm{d}x\right)^{1/p}.$$

This space is characterized as the set of functions in $L^{p^*}(\mathbb{R}^n)$ whose second-order partial derivatives in the distributional sense are in $L^p(\mathbb{R}^n)$. The inclusion $D^{2,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$ is continuous by the Sobolev embedding theorem. Denote by $K = K(n,p)$ the best constant of this embedding, that is,

$$\frac{1}{K(n,p)} = \inf_{u \in D^{2,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\Delta u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}}. \tag{4}$$

Since Lions [26], it is known that the infimum is achieved and that minimizers are positive, radially symmetric decreasing functions, up to translation and multiplication by a nonzero constant. For $p = 2$, it was shown by Edmunds et al. [16] and Lieb [25] that

$$K(n,2) = \frac{16}{n(n-4)(n^2-4)\omega_n^{4/n}},$$

where ω_n denotes the volume of the unit n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$, and that the set of extremal functions is precisely

$$z(x) = c\left(\frac{1}{\lambda + |x - x_0|^2}\right)^{(n-4)/2}, \tag{5}$$

where $\lambda > 0$, $c \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Although the explicit value of $K(n,p)$ and the exact shape of minimizers are not known for $p \neq 2$, the asymptotic behaviors of the extremal functions and their Laplacians were determined by Hulshof and van der Vorst [23] for any $1 < p < n/2$ (see Appendix B).

The first result we prove is the following:

Theorem 1. *Let (M,g) be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$ and $1 < p < n/2$. Then $\mathcal{A}_p^i(M) = K^p$. In particular, given $\varepsilon > 0$, there exists a real constant $B = B(M,g,\varepsilon)$ such that*

$$\|u\|_{L^{p^*}(M)}^p \leqslant \left(K^p + \varepsilon\right)\|\Delta_g u\|_{L^p(M)}^p + B\|u\|_{L^p(M)}^p \tag{6}$$

for all $u \in E_i$.

The proof of this theorem in the case $p = 2$ was based on a partition of unity argument involving harmonic charts and on the Bochner–Lichnerowicz–Weitzenböck integral formula (see [12]). This integral identity is no longer available in the case

$p \neq 2$. In its place, we use Calderon–Zygmund inequalities from the theory of singular integrals and L^p theory of elliptic operators, which demand only standard charts. The case $E_3 = H^{2,p}(M) \cap H_0^{1,p}(M)$ requires an additional result about the sharp Sobolev inequality on bounded Euclidean domains (see Lemma 1).

Concerning the validity of the optimal inequality, contrary to what happens in the first-order case, one cannot hope (2) to hold for $p = 2$, as was shown in [12] for standard spheres of dimension $n \geqslant 6$. We prove the nonvalidity of (2) for $p = 2$ and compact Riemannian manifolds, with or without boundary, which have positive scalar curvature somewhere. More precisely, we have the following:

Theorem 2. *Let (M, g) be a smooth compact n-dimensional Riemannian manifold, with or without boundary, with positive scalar curvature somewhere. Then, the optimal inequality* (2) *is not valid if $n \geqslant 6$ and $p = 2$.*

The proof of this theorem, in the same spirit of Druet in the first-order case [13], depends on knowing the explicit form of the extremal functions. We remark that the optimal second-order Sobolev inequality which includes the first-order term

$$\|u\|_{L^{2^*}(M)}^2 \leqslant K^2 \|\Delta_g u\|_{L^2(M)}^2 + A\|\nabla_g u\|_{L^2(M)}^2 + B\|u\|_{L^2(M)}^2$$

was recently shown by Hebey [22] to be valid on compact Riemannian manifolds without boundary of dimension $n \geqslant 5$.

As a subsequent step, we apply the asymptotically sharp inequality (6) in the study of fourth-order partial differential equations with critical growth on compact Riemannian manifolds, with and without boundary. Specifically, given $a, b, f \in C^0(M)$, if M has no boundary, we seek solutions to the equation:

$$\begin{aligned} &\Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) - \operatorname{div}_g\big(a(x)|\nabla_g u|^{p-2}\nabla_g u\big) + b(x)|u|^{p-2}u \\ &\quad = f(x)|u|^{p^*-2}u \quad \text{in } M, \end{aligned} \tag{P$_1$}$$

and if M has boundary, solutions to the Dirichlet problem:

$$\begin{cases} \Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) - \operatorname{div}_g\big(a(x)|\nabla_g u|^{p-2}\nabla_g u\big) + b(x)|u|^{p-2}u \\ \quad = f(x)|u|^{p^*-2}u \quad \text{in } M, \\ u = \nabla_g u = 0 \qquad \text{on } \partial M, \end{cases} \tag{P$_2$}$$

and to the Navier problem

$$\begin{cases} \Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) - \operatorname{div}_g\big(a(x)|\nabla_g u|^{p-2}\nabla_g u\big) + b(x)|u|^{p-2}u \\ \quad = f(x)|u|^{p^*-2}u \quad \text{in } M, \\ u = \Delta_g u = 0 \qquad \text{on } \partial M. \end{cases} \tag{P$_3$}$$

For $p=2$, Eq. (P_1) appears in conformal geometry. Indeed, given a Riemannian manifold (M,g) of dimension $n\geqslant 5$ with scalar curvature Scal_g and Ricci curvature Ric_g, the following so-called Paneitz–Branson operator is conformally invariant:

$$P_g u=\Delta_g^2 u-\mathrm{div}_g\left(\frac{(n-2)^2+4}{2(n-1)(n-2)}\,\mathrm{Scal}_g\, g-\frac{4}{n-2}\,\mathrm{Ric}_g\right)\mathrm{d}u+\frac{n-4}{2}Q_g u,$$

where

$$Q_g=-\frac{1}{2(n-1)}\Delta_g\,\mathrm{Scal}_g+\frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}\,\mathrm{Scal}_g^2-\frac{2}{(n-2)^2}|\mathrm{Ric}_g|^2.$$

Existence of a conformal metric $\tilde g=u^{4/(n-4)}g$ with scalar curvature $\mathrm{Scal}_{\tilde g}$ and $\mathrm{Ric}_{\tilde g}$ is equivalent to finding a positive solution for the fourth-order equation:

$$P_g u=\frac{n-4}{2}Q_{\tilde g}u^{(n+4)/(n-4)}\quad \text{in } M.$$

When (M,g) is Einstein and $p=2$, this last equation becomes (P_1). Our motivation for investigating (P_1), (P_2) and (P_3) arises from the desire of understanding the role of the geometry in these problems. Problem (P_1) for $p=2$ was studied by Djadli et al. [12], with constant coefficients and special emphasis on spheres, and by Esposito and Robert [17], with subcritical perturbation and more general second-order terms, on compact manifolds.

Nontrivial weak solutions of (P_i) correspond, modulo nonzero constant multiples, to critical points of the functional

$$J(u)=\int_M|\Delta_g u|^p\,\mathrm{d}v_g+\int_M a(x)|\nabla_g u|^p\,\mathrm{d}v_g+\int_M b(x)|u|^p\,\mathrm{d}v_g$$

on the manifold

$$V_i=\left\{u\in E_i\colon \int_M f(x)|u|^{p^*}\,\mathrm{d}v_g=1\right\}.$$

The functional J is said to be coercive on E_i if there exists some positive constant C, dependent only on a and b, such that

$$J(u)\geqslant C\|u\|_{E_i}^p$$

for all $u\in E_i$. This happens, for instance, if $a\geqslant 0$, $b>0$ and M has no boundary, or if $a\geqslant 0$, $b\geqslant 0$ and M has boundary (see Proposition A2 in Appendix A). We say that (H_i) holds if

$$\max_M f>0,\qquad \inf_{V_i}J<\frac{1}{K^p(\max_M f)^{p/p^*}}. \tag{H_i}$$

Under these conditions, we have the following results:

Theorem 3A. *Let (M,g) be a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 3$ and $1 < p < n/2$. Assume that $a, b, f \in C^0(M)$ are such that the functional J is coercive on E_1 and (H_1) holds. Then (P_1) possesses a nontrivial weak solution u. Moreover, if $p = 2$ and $a \in C^{1,\gamma}(M)$, b, $f \in C^\gamma(M)$, then $u \in C^{4,\gamma}(M)$; if, in addition, $f \geqslant 0$ and $a > 0$ is a constant such that $b(x) \leqslant a^2/4$, then (P_1) admits a positive solution.*

Theorem 3B. *Let (M,g) be a smooth compact Riemannian manifold with boundary of dimension $n \geqslant 3$ and $1 < p < n/2$. Assume that $a, b, f \in C^0(M)$ are such that the functional J is coercive on E_2 and (H_2) holds. Then (P_2) possesses a nontrivial weak solution u. Moreover, if $p = 2$ and $a \in C^{1,\gamma}(M)$, b, $f \in C^\gamma(M)$, then $u \in C^{4,\gamma}(M)$.*

Theorem 3C. *Let (M,g) be a smooth compact Riemannian manifold with boundary of dimension $n \geqslant 3$ and $1 < p < n/2$. Assume that $a, b, f \in C^0(M)$ are such that the functional J is coercive on E_3 and (H_3) holds. Then (P_3) possesses a nontrivial weak solution u. Moreover, if b, $f \in C^\gamma(M)$ and either $a \equiv 0$ or $p = 2$ and a is a nonnegative constant, then $u \in C^{4,\gamma}(M)$; if, in addition, $f \geqslant 0$ and $b(x) \leqslant a^2/4$, then (P_3) admits a positive solution.*

Since V_i is not weakly closed in the E_i topology, the direct variational method does not apply. One also encounters difficulties in establishing the regularity of weak solutions, since the Moser iterative scheme fails in our case. The existence part of these theorems is proved through a minimization argument involving Ekeland's variational principle together with a version of the concentration-compactness principle which is a consequence of (6). The argument we use in order to obtain regularity is inspired on the work done by van der Vorst [33] in connection with the biharmonic operator. We remark that the case $p = 2$ and $n \geqslant 7$ of Theorem 3A was proved by Caraffa [10], using the Yamabe method.

An immediate application of the preceding theorems, noticing that

$$u \equiv \left(\int_M f \, \mathrm{d}v_g \right)^{-1/p^*} \in V_i,$$

is the following corollary:

Corollary 1. *Let (M,g) be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$ and $1 < p < n/2$. Assume that $a, b \in C^0(M)$ are such that the functional J is coercive on E_i and $f \in C^0(M)$ is such that $\int_M f \, \mathrm{d}v_g > 0$. If*

$$\left(\frac{\max_M f}{\int_M f \, \mathrm{d}v_g} \right)^{p/p^*} \int_M b \, \mathrm{d}v_g < \frac{1}{K^p},$$

then (P_i) possesses a nontrivial weak solution.

As another application of Theorems 3A–3C, we obtain a result which relates the geometry of the manifold at a point of maximum of f and the behavior of f up to the second order at this point. A version of this result was originally obtained by Druet [14] for the p-Laplacian. The proof involves estimates on the growth of the standard bubbles localized at a maximum point of f, which are obtained from the asymptotic behavior of the minimizers of (4). Fix a positive radially symmetric minimizer $z = z(r)$ for (4). Denote:

$$
\begin{aligned}
I_1 &= I_1(n,p) = \int_{\mathbb{R}^n} z^{p^*}\,\mathrm{d}x, \qquad I_2 = I_2(n,p) = \int_{\mathbb{R}^n} z^{p^*} r^2\,\mathrm{d}x,\\
I_3 &= I_3(n,p) = \int_{\mathbb{R}^n} |\Delta z|^p\,\mathrm{d}x, \qquad I_4^1 = I_4^1(n,p) = \int_{\mathbb{R}^n} |\Delta z|^p r^2\,\mathrm{d}x,\\
I_4^2 &= I_4^2(n,p) = \int_{\mathbb{R}^n} |\Delta z|^{p-1}\big|z'(r)\big| r\,\mathrm{d}x
\end{aligned} \tag{7}
$$

whenever the right-hand side makes sense, and set $I_4 = I_4^1 + 2pI_4^2$. We have:

Corollary 2. *Let (M,g) be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 5$ and $(n+2)/n < p < (n+2)/4$. Let $a \equiv 0$ and $b \in C^0(M)$ be such that the functional J is coercive on E_i. Furthermore, assume that $f \in C^2(M)$, $\max_M f > 0$ and f has a point of maximum x_0 outside the boundary. If*

$$
\frac{\Delta_g f(x_0)}{f(x_0)} > \frac{1}{3}\left(1 - \frac{p^*}{p}\frac{I_1 I_4}{I_2 I_3}\right)\mathrm{Scal}_g(x_0), \tag{8}
$$

then (P_i) possesses a nontrivial weak solution.

We remark that the quotient $(I_1I_4)/(I_2I_3)$ in (8) does not depend on the choice of z.

The methods used above are then applied to the study of the fourth-order Brezis–Nirenberg problem on compact Riemannian manifolds. Indeed, consider the following one-parameter problems:

$$
\Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) = |u|^{p^*-2}u + \lambda|u|^{p-2}u \quad \text{in } M, \tag{BN$_1$}
$$

if M has no boundary,

$$
\begin{cases}
\Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) = |u|^{p^*-2}u + \lambda|u|^{p-2}u & \text{in } M,\\
u = \nabla_g u = 0 & \text{on } \partial M,
\end{cases} \tag{BN$_2$}
$$

and

$$
\begin{cases}
\Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) = |u|^{p^*-2}u + \lambda|u|^{p-2}u & \text{in } M,\\
u = \Delta_g u = 0 & \text{on } \partial M,
\end{cases} \tag{BN$_3$}
$$

if M has boundary. Denote by λ_1 the first eigenvalue associated to the equation

$$\Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) = \lambda|u|^{p-2}u \quad \text{in } M,$$

on E_i. The variational characterization of λ_1 is given by

$$\lambda_1 = \inf_{u\in E_i\setminus\{0\}} \frac{\int_M |\Delta_g u|^p \,\mathrm{d}v_g}{\int_M |u|^p \,\mathrm{d}v_g}.$$

Clearly, $\lambda_1 = 0$ on E_1 and $\lambda_1 > 0$ on E_2 and on E_3.

In the spirit of Brezis and Nirenberg [9], we are interested in determining the range of values of λ for which (BN_1), (BN_2) and (BN_3) admit nontrivial solutions. With the aid of an eigenfunction associated to λ_1, it is always possible to find nontrivial solutions for λ close to λ_1. A more difficult task is to obtain solutions for λ far from λ_1. For $p = 2$ and in Euclidean bounded domains of dimension $n \geqslant 8$, Edmunds, Fortunato and Janelli [16] and van der Vorst [34] established, respectively, the existence of nontrivial solutions of (BN_2) and the existence of positive solutions of (BN_3) for any $0 < \lambda < \lambda_1$. In addition, still in this context, it is known that (BN_2) has no nontrivial solutions for $\lambda < 0$ and (BN_3) has no positive solutions for $\lambda \leqslant 0$ in star-shaped domains, and that (BN_3) has no positive solutions for $\lambda \geqslant \lambda_1$ (see [28,30]).

We show that the situation changes drastically when we consider compact Riemannian manifolds with boundary which have positive scalar curvature somewhere (similar phenomena occur in the second order Brezis–Nirenberg problem; see [7] and the references therein). Indeed, in this case, for $n \geqslant 6$ we establish the existence of nontrivial solutions for (BN_2) and (BN_3) for any $\lambda < \lambda_1$ and of positive solutions for (BN_3) for $0 \leqslant \lambda < \lambda_1$. In particular, the existence of nontrivial solutions to (BN_2) for $\lambda < 0$ and of positive solutions to (BN_3) for $\lambda = 0$ contrasts with the results mentioned above for star-shaped Euclidean domains. Our results seem to point to the existence of only one critical dimension $n = 5$ in the case of manifolds with positive scalar curvature somewhere, in comparison with the Euclidean case, where $n = 5, 6, 7$ are the critical dimensions (see [31]). An analogous version of these results is proved on compact Riemannian manifolds without boundary in the case $p = 2$. Moreover, we also discuss (BN_2) and (BN_3) for other values of p on compact manifolds of dimension $n \geqslant 6$ which are flat on a neighborhood, which include bounded domains in $\mathbb{R}^n$. Nontrivial solutions are found for $n/(n-2) < p \leqslant \sqrt{n/2}$ and $0 < \lambda < \lambda_1$. This generalizes Theorem 1.1 of [16] and Theorem 3 of [34]. These results are resumed in the following theorems:

Theorem 4A. *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 6$. If $p = 2$ and M has positive scalar curvature somewhere, then* (BN_1) *has a nontrivial solution in $C^{4,\gamma}(M)$ for any $\lambda < \lambda_1$. If $\lambda \geqslant \lambda_1$, then* ($BN_1$) *has no positive solution.*

Theorem 4B. *Let (M, g) be a smooth compact Riemannian manifold with boundary of dimension $n \geqslant 6$. Then*:

(i) *If $p = 2$ and M has positive scalar curvature somewhere,* (BN_2) *has a nontrivial solution $C^{4,\gamma}(M)$ for any $\lambda < \lambda_1$.*
(ii) *If $n/(n-2) < p \leqslant \sqrt{n/2}$ and M is flat in a neighborhood,* (BN_2) *has a nontrivial solution in $C^{4,\gamma}(M)$ for any $0 < \lambda < \lambda_1$.*

Theorem 4C. *Let (M, g) be a smooth compact Riemannian manifold with boundary of dimension $n \geqslant 6$. Then*:

(i) *If $p = 2$ and M has positive scalar curvature somewhere,* (BN_3) *has a positive solution for any $0 \leqslant \lambda < \lambda_1$ and a nontrivial solution for any $\lambda < 0$ in $C^{4,\gamma}(M)$. If $\lambda \geqslant \lambda_1$, then* ($BN_3$) *has no positive solution.*
(ii) *If $n/(n-2) < p \leqslant \sqrt{n/2}$ and M is flat in a neighborhood,* (BN_3) *has a positive solution in $C^{4,\gamma}(M)$ for any $0 < \lambda < \lambda_1$. If $\lambda \geqslant \lambda_1$, then* ($BN_3$) *has no positive solution.*

The arguments utilized in the proof of these results again are based on the minimization technique and estimates of the growth of standard bubbles. In the case $p = 2$, the more precise estimates are used. Theorem 4A was proved in [10] for $n > 6$; in fact, Caraffa considered a more general equation than (BN_1) and obtained a sharper result.

The structure of the paper is as follows. In Section 2 we prove the asymptotically sharp Sobolev inequality. In Section 3, we prove that this is the best we can have for $p = 2$ for manifolds with positive scalar curvature somewhere. In Section 4 we prove Theorems 3A–3C and Corollary 2, and in Section 5 we consider the fourth-order Brezis–Nirenberg problem, proving Theorems 4A–4C.

2. The asymptotically sharp Sobolev inequality

The proof of Theorem 1 will follow from Propositions 1 and 2 below.

Proposition 1. *Let (M, g) be a compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$ and $1 < p < n/2$. Let $A, B \in \mathbb{R}$ be such that*

$$\|u\|^p_{L^{p^*}(M)} \leqslant A\|\Delta_g u\|^p_{L^p(M)} + B\|u\|^p_{L^p(M)}$$

for all $u \in E_i$. Then $A \geqslant K^p$.

Proof. We proceed by contradiction. Assume that there exist $A < K^p$ and $B \in \mathbb{R}$ such that the above inequality is true for all $u \in E_i$. Fix $x_0 \in M \backslash \partial M$ and a geodesic ball $B_\delta(x_0)$, where $\delta > 0$ will be chosen later. Considering a normal coordinates system defined on $B_\delta(x_0)$, we have:

$$\left|g^{ij} - \delta_{ij}\right| \leqslant \varepsilon_1, \qquad \left|\Gamma^k_{ij}\right| \leqslant \varepsilon_1$$

and

$$(1 - \varepsilon_1)\,\mathrm{d}x \leqslant \mathrm{d}v_g \leqslant (1 + \varepsilon_1)\,\mathrm{d}x,$$

for some $\varepsilon_1 > 0$ that can be chosen as small as we wish, provided we take δ small enough. In the sequel, we will denote by ε_j several possibly different positive constants independent of δ. Denoting by B_δ the Euclidean ball of center 0 and radius δ, it follows that for any $u \in C_0^\infty(B_\delta)$ we have:

$$\begin{aligned}
\left(\int_{B_\delta} |u|^{p^*}\,\mathrm{d}x\right)^{p/p^*} &\leqslant \left(\frac{1}{1-\varepsilon_1}\int_M |u|^{p^*}\,\mathrm{d}v_g\right)^{p/p^*} \\
&\leqslant (1+\varepsilon_2)A\int_M |\Delta_g u|^p\,\mathrm{d}v_g + (1+\varepsilon_2)B\int_M |u|^p\,\mathrm{d}v_g \\
&\leqslant (1+\varepsilon_3)A\int_{B_\delta} |\Delta_g u|^p\,\mathrm{d}x + (1+\varepsilon_3)B\int_{B_\delta} |u|^p\,\mathrm{d}x, \qquad (9)
\end{aligned}$$

for some positive numbers $\varepsilon_2, \varepsilon_3 = \mathrm{O}(\varepsilon_1)$. Writing

$$\Delta_g u = \Delta u + \sum_{i,j=1}^{n} \left(g^{ij} - \delta_{ij}\right)\partial_{ij} u + \sum_{i,j,k=1}^{n} g^{ij}\Gamma_{ij}^k \partial_k u, \qquad (10)$$

and using the elementary inequality $(a+b)^p \leqslant (1+\varepsilon_4)a^p + C_{\varepsilon_4} b^p$, where ε_4 will be chosen later, we find

$$\int_{B_\delta} |\Delta_g u|^p\,\mathrm{d}x \leqslant (1+\varepsilon_4)\int_{B_\delta} |\Delta u|^p\,\mathrm{d}x + \varepsilon_1^p C_{\varepsilon_4}\int_{B_\delta} \left|\partial^2 u\right|^p\,\mathrm{d}x + \varepsilon_1^p C_{\varepsilon_4}\int_{B_\delta} |\partial u|^p\,\mathrm{d}x. \qquad (11)$$

By the Calderon–Zygmund inequality (see [18]), there exists a positive constant $C_{n,p}$, dependent only on n and p, such that

$$\int_{B_\delta} \left|\partial^2 u\right|^p\,\mathrm{d}x \leqslant C_{n,p}\int_{B_\delta} |\Delta u|^p\,\mathrm{d}x, \qquad (12)$$

while interpolation of lower-order derivatives yields

$$\int_{B_\delta} |\partial u|^p\,\mathrm{d}x \leqslant \varepsilon_5\int_{B_\delta} \left|\partial^2 u\right|^p\,\mathrm{d}x + C_{\varepsilon_5,\delta}\int_{B_\delta} |u|^p\,\mathrm{d}x \qquad (13)$$

for any $\varepsilon_5 > 0$, where, with respect to δ, we have $C_{\varepsilon_5,\delta} = \mathrm{O}(\delta^{-p})$. Therefore, putting together (9), (11)–(13), and choosing $\varepsilon_1, \varepsilon_4$ and ε_5 sufficiently small, we find $\delta > 0$ such that for all $u \in C_0^\infty(B_\delta)$ there holds

$$\left(\int_{B_\delta} |u|^{p^*}\,\mathrm{d}x\right)^{p/p^*} \leqslant A_1\int_{B_\delta} |\Delta u|^p\,\mathrm{d}x + B_{1,\delta}\int_{B_\delta} |u|^p\,\mathrm{d}x \qquad (14)$$

for some real numbers $A_1 < K^p$ and $B_{1,\delta} = \mathrm{O}(\delta^{-p})$. On the other hand, by Hölder's inequality,

$$\int_{B_\delta} |u|^p \,\mathrm{d}x \leqslant |B_\delta|^{2p/n} \left(\int_{B_\delta} |u|^{p^*} \,\mathrm{d}x \right)^{p/p^*},$$

where $|B_\delta|$ stands for the Euclidean volume of B_δ. Thus, choosing δ small enough so that $|B_\delta|^{2p/n} B_{1,\delta} < 1$ and

$$\frac{A_1}{1 - |B_\delta|^{2p/n} B_{1,\delta}} < K^p,$$

it follows that there exists $A_2 < K^p$ such that for all $u \in C_0^\infty(B_\delta)$ there holds:

$$\left(\int_{B_\delta} |u|^{p^*} \,\mathrm{d}x \right)^{p/p^*} \leqslant A_2 \int_{B_\delta} |\Delta u|^p \,\mathrm{d}x. \tag{15}$$

Now, given $u \in C_0^\infty(\mathbb{R}^n)$ and $\varepsilon > 0$, define $u_\varepsilon(x) = \varepsilon^{-n/p^*} u(x/\varepsilon)$. For ε small enough, we have $u_\varepsilon \in C_0^\infty(B_\delta)$, and so

$$\left(\int_{\mathbb{R}^n} |u_\varepsilon|^{p^*} \,\mathrm{d}x \right)^{p/p^*} \leqslant A_2 \int_{\mathbb{R}^n} |\Delta u_\varepsilon|^p \,\mathrm{d}x.$$

Since this is precisely the rescaling such that

$$\|u_\varepsilon\|_{L^{p^*}(\mathbb{R}^n)} = \|u\|_{L^{p^*}(\mathbb{R}^n)}$$

and

$$\|\Delta u_\varepsilon\|_{L^p(\mathbb{R}^n)} = \|\Delta u\|_{L^p(\mathbb{R}^n)},$$

we conclude that

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} \,\mathrm{d}x \right)^{p/p^*} \leqslant A_2 \int_{\mathbb{R}^n} |\Delta u|^p \,\mathrm{d}x$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, contradicting the definition of K. □

The proof of Proposition 2 in the case $E_3 = H^{2,p}(M) \cap H_0^{1,p}(M)$ requires the following lemma on the Euclidean sharp second-order Sobolev inequality. For $p = 2$, this result was obtained by van der Vorst [33] using the concentration-compactness principle, Talenti's comparison principle and a Pohozaev type identity. Our proof simplifies his argument for any $1 < p < n/2$.

Lemma 1. *Let Ω be a bounded open set of $\mathbb{R}^n$ with smooth boundary, $n \geqslant 3$ and $1 < p < n/2$. Then*

$$\|u\|_{L^{p^*}(\Omega)} \leqslant K \|\Delta u\|_{L^p(\Omega)} \tag{16}$$

for every $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$. Moreover, K is the best constant in this inequality.

Proof. Denote by $K(\Omega)$ the best constant in the embedding of $H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$, i.e.,

$$\frac{1}{K(\Omega)} = \inf_{\substack{u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega) \\ u \neq 0}} \frac{\|\Delta u\|_{L^p(\Omega)}}{\|u\|_{L^{p^*}(\Omega)}}.$$

Proposition 1 implies that $K(\Omega) \geqslant K$. Assume by contradiction that $K(\Omega) > K$. Since the set $\{u \in C^2(\overline{\Omega}):\ u = 0 \text{ on } \partial\Omega\}$ is dense in $H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$, it follows that there exists $u \in C^2(\overline{\Omega})$ in this set such that

$$\frac{\|\Delta u\|_{L^p(\Omega)}}{\|u\|_{L^{p^*}(\Omega)}} < \frac{1}{K}.$$

Set

$$f(x) = \begin{cases} -|\Delta u| & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^n \backslash \Omega, \end{cases}$$

and define

$$w = G * f,$$

where $*$ and G denote, respectively, the convolution operation and the Green function of the Laplacian operator in $\mathbb{R}^n$. From the Hardy–Littlewood–Sobolev inequality (see [25]) and Calderon–Zygmund estimates for singular integrals (see [18]), it follows that $w \in D^{2,p}(\mathbb{R}^n) \cap C^{1,\gamma}(\mathbb{R}^n)$ and verifies

$$\Delta w = f \quad \text{in } \mathbb{R}^n.$$

Moreover, since G is a strictly negative function, we have $w > 0$ in $\mathbb{R}^n$. As

$$\Delta(w \pm u) \leqslant 0 \quad \text{in } \Omega, \qquad w \pm u > 0 \quad \text{on } \partial\Omega,$$

the maximum principle provides us $w > |u|$ in Ω. Therefore,

$$\|\Delta w\|_{L^p(\mathbb{R}^n)} = \|\Delta u\|_{L^p(\Omega)}$$

and

$$\|w\|_{L^{p^*}(\mathbb{R}^n)} > \|u\|_{L^{p^*}(\Omega)}$$

whence

$$\frac{\|\Delta w\|_{L^p(\mathbb{R}^n)}}{\|w\|_{L^{p^*}(\mathbb{R}^n)}} < \frac{1}{K},$$

a contradiction. □

Remarks. (1) Since $C_0^\infty(\Omega)$ is dense in $H_0^{2,p}(\Omega)$ and zero extensions of functions in $H_0^{2,p}(\Omega)$ belong to $D^{2,p}(\mathbb{R}^n)$, one concludes directly from a scaling argument that Lemma 1 also holds for $H_0^{2,p}(\Omega)$ in place of $H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$.

(2) Using the Talenti comparison principle [32], a Pohozaev type identity for elliptic systems [28] and the regularity results of Section 4.3, in the same spirit of [33] one proves that the best constant K is never attained in $H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$.

Proposition 2. *Let (M, g) be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$ and $1 < p < n/2$. Then, given $\varepsilon > 0$, there exists a real constant $B = B(M, g, \varepsilon)$ such that*

$$\|u\|_{L^{p^*}(M)}^p \leqslant (K^p + \varepsilon)\|\Delta_g u\|_{L^p(M)}^p + B\|u\|_{L^p(M)}^p$$

for all $u \in E_i$.

Proof. Let $\varepsilon > 0$ be given. We will denote by ε_j several possibly different positive constants independent of δ. For some $\delta > 0$ small enough to be determined later, let $\{B_k\}_{k=1,\dots,N_\delta}$ be a finite covering of M by geodesic balls of radius δ such that, in normal geodesic coordinates in each of these balls, we have

$$\left|g^{ij} - \delta_{ij}\right| \leqslant \varepsilon_1, \qquad \left|\Gamma_{ij}^k\right| \leqslant \varepsilon_1$$

and

$$(1 - \varepsilon_1)\,\mathrm{d}x \leqslant \mathrm{d}v_g \leqslant (1 + \varepsilon_1)\,\mathrm{d}x$$

for some $\varepsilon_1 > 0$ that can be chosen as small as we wish, provided we take δ small enough. Let $\{\phi_k\}_{k=1,\dots,N_\delta}$ be a partition of unity subordinated to the covering $\{B_k\}$ such that $\phi_k^{1/p} \in C_0^2(B_k)$ for each k. First, we write

$$\|u\|_{L^{p^*}(M)}^p = \left\|\sum_k \phi_k |u|^p\right\|_{L^{p^*/p}(M)} \leqslant \sum_k \|\phi_k |u|^p\|_{L^{p^*/p}(M)} = \sum_k \left\|\phi_k^{1/p} u\right\|_{L^{p^*}(M)}^p$$

$$\leqslant (1+\varepsilon_1)^{p/p^*}\sum_k\biggl(\int\limits_{B_k}\phi_k^{p^*/p}|u|^{p^*}\,\mathrm{d}x\biggr)^{p/p^*}. \tag{17}$$

Then, decomposing $\Delta_g(\phi_k^{1/p}u)$ as in (10), using the elementary inequality $(1-\varepsilon_2)a^p \leqslant (a+b)^p + C_{\varepsilon_2}b^p$, where we choose $\varepsilon_2 = \mathrm{O}(\varepsilon_1)$ small, the Calderon–Zygmund and the interpolation inequalities (12) and (13), with $\phi_k^{1/p}u$ in place of u, we find:

$$\begin{aligned}
\int\limits_M \bigl|\Delta_g\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}v_g &\geqslant (1-\varepsilon_1)\int\limits_{B_k}\bigl|\Delta_g\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}x \\
&\geqslant (1-\varepsilon_1)(1-\varepsilon_2)\int\limits_{B_k}\bigl|\Delta\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}x - \varepsilon_1^p C_{\varepsilon_2}\int\limits_{B_k}\bigl|\partial^2\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}x \\
&\quad - \varepsilon_1^p C_{\varepsilon_2}\int\limits_{B_k}\bigl|\partial\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}x \\
&\geqslant (1-\varepsilon_3)\int\limits_{B_k}\bigl|\Delta\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}x - C_{\varepsilon_1,\delta}\int\limits_{B_k}\phi_k|u|^p\,\mathrm{d}x,
\end{aligned}$$

with $\varepsilon_3 = \mathrm{O}(\varepsilon_1)$ a positive small number. Noticing that $\phi_k^{1/p}u \in H_0^{2,p}(B_k)$, if $u \in E_1$ or $u \in E_2$, and $u \in H^{2,p}(B_k)\cap H_0^{1,p}(B_k)$ if $u \in E_3$, Lemma 1 implies:

$$\int\limits_M \bigl|\Delta_g\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}v_g \geqslant \frac{1-\varepsilon_3}{K^p}\biggl(\int\limits_{B_k}\phi_k^{p^*/p}|u|^{p^*}\,\mathrm{d}x\biggr)^{p/p^*} - C_{\varepsilon_1,\delta}\int\limits_{B_k}\phi_k|u|^p\,\mathrm{d}x. \tag{18}$$

Putting together (17) and (18), and applying again the elementary inequality $(a+b)^p \leqslant (1+\varepsilon_4)a^p + C_{\varepsilon_4}b^p$, choosing $\varepsilon_4 = \mathrm{O}(\varepsilon_1)$ small, we obtain

$$\begin{aligned}
\|u\|_{L^{p^*}(M)}^p &\leqslant (1+\varepsilon_5)K^p\sum_k\int\limits_M\bigl|\Delta_g\bigl(\phi_k^{1/p}u\bigr)\bigr|^p\,\mathrm{d}v_g + C_{\varepsilon_1,\delta}\int\limits_M|u|^p\,\mathrm{d}v_g \\
&\leqslant (1+\varepsilon_6)K^p\int\limits_M|\Delta_g u|^p\,\mathrm{d}v_g + C_{\varepsilon_1}\sum_k\int\limits_M\bigl|\nabla_g\bigl(\phi_k^{1/p}\bigr)\bigr|^p|\nabla_g u|^p\,\mathrm{d}v_g \\
&\quad + C(\varepsilon_1)\sum_k\int\limits_M\bigl|\Delta_g\bigl(\phi_k^{1/p}\bigr)\bigr|^p|u|^p\,\mathrm{d}v_g + C_{\varepsilon_1,\delta}\int\limits_M|u|^p\,\mathrm{d}v_g \\
&\leqslant (1+\varepsilon_6)K^p\int\limits_M|\Delta_g u|^p\,\mathrm{d}v_g + C_{\varepsilon_1,\delta}\int\limits_M|\nabla_g u|^p\,\mathrm{d}v_g + C_{\varepsilon_1,\delta}\int\limits_M|u|^p\,\mathrm{d}v_g
\end{aligned}$$

for some positive numbers $\varepsilon_5, \varepsilon_6 = \mathrm{O}(\varepsilon_1)$, since $|\nabla_g(\phi_k^{1/p})|$ and $|\Delta_g(\phi_k^{1/p})|$ are both bounded by a constant C depending on δ. Choosing ε_1 sufficiently small and correspondingly fixing $\delta > 0$, we get:

$$\|u\|^p_{L^{p^*}(M)} \leqslant \left(K^p + \frac{\varepsilon}{2}\right)\|\Delta_g u\|^p_{L^p(M)} + \widetilde{C}_{\varepsilon_1,\delta}\|\nabla_g u\|^p_{L^p(M)} + \widetilde{C}_{\varepsilon_1,\delta}\|u\|^p_{L^p(M)}. \quad (19)$$

On the other hand, by the L^p-theory of linear elliptic operators, there exists a positive constant $C_1(\delta)$ such that

$$\begin{aligned}\int_{B_k} \left|\partial^2\left(\phi_k^{1/p} u\right)\right|^p \mathrm{d}x &\leqslant C_1(\delta) \int_{B_k} \left|\Delta_g\left(\phi_k^{1/p} u\right)\right|^p \mathrm{d}x \\ &\leqslant C_2(\delta)\left(\int_{B_k} |\Delta_g u|^p \,\mathrm{d}x + \int_{B_k} |\nabla_g u|^p \,\mathrm{d}x + \int_{B_k} |u|^p \,\mathrm{d}x\right) \\ &\leqslant \frac{C_2(\delta)}{1-\varepsilon_1}\left(\|\Delta_g u\|^p_{L^p(M)} + \|\nabla_g u\|^p_{L^p(M)} + \|u\|^p_{L^p(M)}\right). \end{aligned} \quad (20)$$

Using again the interpolation inequality of lower-order derivatives

$$\int_{B_k} \left|\partial\left(\phi_k^{1/p} u\right)\right|^p \mathrm{d}x \leqslant \theta \int_{B_k} \left|\partial^2\left(\phi_k^{1/p} u\right)\right|^p \mathrm{d}x + C_{\theta,\delta} \int_{B_k} \phi_k |u|^p \,\mathrm{d}x,$$

since $|\nabla_g u| \leqslant (1+\varepsilon_1)|\partial u|$, it follows that

$$\begin{aligned}\|\nabla_g u\|^p_{L^p(M)} &= \sum_k \left\|\phi_k^{1/p}\nabla_g u\right\|^p_{L^p(M)} \leqslant (1+\varepsilon_1)^{p+1} \sum_k \int_{B_k} \left|\phi_k^{1/p}\partial u\right|^p \mathrm{d}x \\ &\leqslant (1+\varepsilon_7)\sum_k \int_{B_k} \left|\partial\left(\phi_k^{1/p} u\right)\right|^p \mathrm{d}x + C_{\varepsilon_1}\sum_k \int_{B_k} \left|\partial\left(\phi_k^{1/p}\right)\right|^p |u|^p \,\mathrm{d}x \\ &\leqslant \theta(1+\varepsilon_7)\sum_k \int_{B_k} \left|\partial^2\left(\phi_k^{1/p} u\right)\right|^p \mathrm{d}x + C_{\varepsilon_1,\delta,\theta}\|u\|^p_{L^p(M)}, \end{aligned} \quad (21)$$

where $\varepsilon_7 = \mathrm{O}(\varepsilon_1)$. Thus, choosing θ small enough, we obtain from (20) and (21),

$$\frac{1}{2}\|\nabla_g u\|^p_{L^p(M)} \leqslant \frac{\varepsilon}{4\widetilde{C}_{\varepsilon_1,\delta}}\|\Delta_g u\|^p_{L^p(M)} + C_{\varepsilon_1,\delta,\theta}\|u\|^p_{L^p(M)}. \quad (22)$$

Finally, coupling (19) with (22), we find $B > 0$ depending only on M, g and ε such that

$$\|u\|^p_{L^{p^*}(M)} \leqslant (K^p + \varepsilon)\|\Delta_g u\|^p_{L^p(M)} + B\|u\|^p_{L^p(M)}$$

for all $u \in E_i$. $\square$

3. The nonvalidity of the optimal inequality

Proof of Theorem 2. In order to prove this theorem, we construct a family of functions $(u_\varepsilon) \subset C_0^\infty(M)$ such that

$$\frac{\|u_\varepsilon\|^2_{L^{2^*}(M)} - K^2\|\Delta_g u_\varepsilon\|^2_{L^2(M)}}{\|u_\varepsilon\|^2_{L^2(M)}} \to +\infty$$

as ε approaches zero. Fix $x_0 \in M\backslash\partial M$ such that $\mathrm{Scal}_g(x_0) > 0$ and a geodesic ball $B_\delta(x_0) \subset M\backslash\partial M$. Consider a radial cutoff function $\eta \in C^\infty(B_\delta)$ satisfying $\eta = 1$ in $B_{\delta/2}$, $\eta = 0$ outside B_δ and $0 \leqslant \eta \leqslant 1$ in B_δ. Define, up to the exponential chart $\exp_{x_0}$,

$$u_\varepsilon(x) = \eta(x) z_\varepsilon(x),$$

where

$$z_\varepsilon(x) = \varepsilon^{-n/2^*} z\left(\frac{x}{\varepsilon}\right) \quad \text{with } z(x) = \frac{1}{(1+|x|^2)^{(n-4)/2}}$$

being an extremal function for the Sobolev quotient (4) in $D^{2,2}(\mathbb{R}^n)$. In particular,

$$\|z\|^2_{L^{2^*}(\mathbb{R}^n)} = K^2\|\Delta z\|^2_{L^2(\mathbb{R}^n)}. \tag{23}$$

We will estimate the asymptotic behavior of $\|u_\varepsilon\|^2_{L^{2^*}(M)}$, $\|u_\varepsilon\|^2_{L^2(M)}$ and $\|\Delta u_\varepsilon\|^2_{L^2(M)}$ with respect to ε near the origin. The result of these computations will involve the scalar curvature $\mathrm{Scal}_g(x_0)$ and the constants I_1, I_2, I_3 and I_4 introduced in (7).

(1) Estimate of $\|u_\varepsilon\|^2_{L^{2^*}(M)}$.

Write $\eta^{2^*}(x) = 1 + \mathrm{O}(r^3)$ and use the expansion of the metric in normal geodesic coordinates up to the third order in order to obtain (see [19])

$$\sqrt{\det g} = 1 - \frac{1}{6}\sum_{i,j=1}^n \mathrm{Ric}_{ij}(x_0)x_i x_j + \mathrm{O}\big(r^3\big), \tag{24}$$

where Ric_{ij} denotes the components of the Ricci tensor in these coordinates. Then,

$$\begin{aligned}\int_M u_\varepsilon^{2^*}\,\mathrm{d}v_g &= \int_{B_\delta} z_\varepsilon^{2^*}\,\mathrm{d}x - \frac{1}{6}\sum_{i,j=1}^n \mathrm{Ric}_{ij}(x_0)\int_{B_\delta} z_\varepsilon^{2^*} x_i x_j\,\mathrm{d}x + \int_{B_\delta} z_\varepsilon^{2^*}\mathrm{O}\big(r^3\big)\,\mathrm{d}x \\ &= \int_{B_\delta} z_\varepsilon^{2^*}\,\mathrm{d}x - \frac{\mathrm{Scal}_g(x_0)}{6n}\int_{B_\delta} z_\varepsilon^{2^*} r^2\,\mathrm{d}x + \int_{B_\delta} z_\varepsilon^{2^*}\mathrm{O}\big(r^3\big)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} z^{2^*}\,\mathrm{d}x - \int_{\mathbb{R}^n\setminus B_{\delta/\varepsilon}} z^{2^*}\,\mathrm{d}x - \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 \int_{\mathbb{R}^n} z^{2^*} r^2\,\mathrm{d}x \\
&\quad + \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 \int_{\mathbb{R}^n\setminus B_{\delta/\varepsilon}} z^{2^*} r^2\,\mathrm{d}x + \varepsilon^3 \int_{B_{\delta/\varepsilon}} z^{2^*}\mathrm{O}\big(r^3\big)\,\mathrm{d}x.
\end{aligned}
$$

After a straightforward computation, we find for any $n \geqslant 5$ that

$$
\|u_\varepsilon\|^2_{L^{2^*}(M)} = \|z\|^2_{L^{2^*}(\mathbb{R}^n)} - \frac{2}{2^*}\frac{\mathrm{Scal}_g(x_0)}{6n} I_2\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big). \tag{25}
$$

(2) Estimate of $\|u_\varepsilon\|^2_{L^2(M)}$.
In this case, we write:

$$
\int_M u_\varepsilon^2\,\mathrm{d}v_g = \mathrm{O}(1)\int_{B_\delta} z_\varepsilon^2\,\mathrm{d}x = \mathrm{O}\big(\varepsilon^4\big)\int_{B_{\delta/\varepsilon}} z^2\,\mathrm{d}x
$$

and obtain by direct computation,

$$
\|u_\varepsilon\|^2_{L^2(M)} = \begin{cases} \mathrm{O}\big(\varepsilon^2\big) & \text{if } n = 6, \\ \mathrm{O}\big(\varepsilon^3\big) & \text{if } n = 7, \\ \mathrm{O}\big(\varepsilon^4|\ln\varepsilon|\big) & \text{if } n = 8, \\ \mathrm{O}\big(\varepsilon^4\big) & \text{if } n \geqslant 9. \end{cases} \tag{26}
$$

(3) Estimate of $\|\Delta_g u_\varepsilon\|^2_{L^2(M)}$.
First, write

$$
\begin{aligned}
\int_M |\Delta_g u_\varepsilon|^2\,\mathrm{d}v_g &= \int_{B_\delta} \big|\eta\Delta_g z_\varepsilon + 2\langle\nabla_g\eta, \nabla_g z_\varepsilon\rangle + (\Delta_g\eta)z_\varepsilon\big|^2\,\mathrm{d}v_g \\
&= \int_{B_{\delta/2}} |\Delta_g z_\varepsilon|^2\,\mathrm{d}v_g + \int_{B_\delta\setminus B_{\delta/2}} \big|\eta\Delta_g z_\varepsilon + 2\langle\nabla_g\eta, \nabla_g z_\varepsilon\rangle + (\Delta_g\eta)z_\varepsilon\big|^2\,\mathrm{d}v_g.
\end{aligned} \tag{27}
$$

In order to compute the first term of the right-hand side of (27), we write the Laplacian in normal geodesic coordinates and, noticing that $\Delta z_\varepsilon(r) < 0$ and $z_\varepsilon'(r) < 0$ for $r > 0$, one has

$$
\begin{aligned}
|\Delta_g z_\varepsilon|^2 &= \big|\Delta z_\varepsilon + z_\varepsilon'(r)\partial_r\big(\ln\sqrt{\det g}\,\big)\big|^2 \\
&= |\Delta z_\varepsilon|^2 + 2|\Delta z_\varepsilon|\big|z_\varepsilon'(r)\big|\partial_r\big(\ln\sqrt{\det g}\,\big) + \big|z_\varepsilon'(r)\partial_r\big(\ln\sqrt{\det g}\,\big)\big|^2.
\end{aligned}
$$

From (24), there follows that

$$\partial_r\big(\ln\sqrt{\det g}\,\big) = -\frac{1}{\sqrt{\det g}}\frac{1}{3}\sum_{i,j=1}^{n}\mathrm{Ric}_{ij}(x_0)\frac{x_i x_j}{r} + \mathrm{O}\big(r^2\big).$$

Therefore, through standard computations, we obtain, for $n \geqslant 7$,

$$\begin{aligned}\int\limits_{B_{\delta/2}} |\Delta z_\varepsilon|^2\,\mathrm{d}v_g &= \int\limits_{\mathbb{R}^n} |\Delta z|^2\,\mathrm{d}x - \int\limits_{\mathbb{R}^n\setminus B_{\delta/(2\varepsilon)}} |\Delta z|^2\,\mathrm{d}x - \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2\int\limits_{\mathbb{R}^n}|\Delta z|^2 r^2\,\mathrm{d}x \\ &\quad + \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2\int\limits_{\mathbb{R}^n\setminus B_{\delta/(2\varepsilon)}}|\Delta z|^2 r^2\,\mathrm{d}x + \varepsilon^3\int\limits_{B_{\delta/(2\varepsilon)}}|\Delta z|^2\mathrm{O}\big(r^3\big)\,\mathrm{d}x \\ &= \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - \frac{\mathrm{Scal}_g(x_0)}{6n}I_4^1\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big),\end{aligned}$$

$$\begin{aligned}&\int\limits_{B_{\delta/2}} |\Delta z_\varepsilon|\big|z_\varepsilon'(r)\big|\partial_r\big(\ln\sqrt{\det g}\,\big)\,\mathrm{d}v_g \\ &\quad = -\frac{\mathrm{Scal}_g(x_0)}{3n}\int\limits_{B_{\delta/2}}|\Delta z_\varepsilon|\big|z_\varepsilon'(r)\big|r\,\mathrm{d}x + \int\limits_{B_{\delta/2}}|\Delta z_\varepsilon|\big|z_\varepsilon'(r)\big|\mathrm{O}\big(r^2\big)\,\mathrm{d}x \\ &\quad = -\frac{\mathrm{Scal}_g(x_0)}{3n}\varepsilon^2\int\limits_{\mathbb{R}^n}|\Delta z|\big|z'(r)\big|r\,\mathrm{d}x + \frac{\mathrm{Scal}_g(x_0)}{3n}\varepsilon^2\int\limits_{\mathbb{R}^n\setminus B_{\delta/(2\varepsilon)}}|\Delta z|\big|z'(r)\big|r\,\mathrm{d}x \\ &\qquad + \varepsilon^3\int\limits_{B_{\delta/(2\varepsilon)}}|\Delta z|\big|z'(r)\big|\mathrm{O}\big(r^2\big)\,\mathrm{d}x \\ &\quad = -\frac{\mathrm{Scal}_g(x_0)}{3n}I_4^2\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big),\end{aligned}$$

and

$$\int\limits_{B_{\delta/2}}\big|z_\varepsilon'(r)\partial_r\big(\ln\sqrt{\det g}\,\big)\big|^2\,\mathrm{d}v_g = \varepsilon^4\int\limits_{B_{\delta/(2\varepsilon)}}\big|z'(r)\big|^2\mathrm{O}\big(r^2\big)\,\mathrm{d}x = \mathrm{o}\big(\varepsilon^2\big).$$

If $n=6$, we have:

$$\int_{B_{\delta/2}} |\Delta z_\varepsilon|^2 \,\mathrm{d}v_g = \int_{\mathbb{R}^n} |\Delta z|^2 \,\mathrm{d}x - \int_{\mathbb{R}^n\setminus B_{\delta/(2\varepsilon)}} |\Delta z|^2 \,\mathrm{d}x - \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 \int_{B_{\delta/(2\varepsilon)}} |\Delta z|^2 r^2 \,\mathrm{d}x$$

$$+\varepsilon^3 \int_{B_{\delta/(2\varepsilon)}} |\Delta z|^2 \mathrm{O}\big(r^3\big)\,\mathrm{d}x$$

$$= \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - \omega_{n-1}(n-4)^2 \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 \int_0^{\delta/(2\varepsilon)} \frac{(6+2s^2)^2}{(1+s^2)^6} s^7 \,\mathrm{d}s + \mathrm{O}\big(\varepsilon^2\big)$$

$$= \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - \omega_{n-1}(n-4)^2 \frac{2\,\mathrm{Scal}_g(x_0)}{3n} \frac{\varepsilon^2|\ln\varepsilon|}{(\varepsilon^2+1)^3} + \mathrm{O}\big(\varepsilon^2\big),$$

$$\int_{B_{\delta/2}} |\Delta z_\varepsilon|\big|z_\varepsilon'(r)\big|\partial_r\big(\ln\sqrt{\det g}\,\big)\,\mathrm{d}v_g$$

$$= -\frac{\mathrm{Scal}_g(x_0)}{3n}\varepsilon^2 \int_{B_{\delta/(2\varepsilon)}} |\Delta z|\big|z'(r)\big| r \,\mathrm{d}x + \varepsilon^3 \int_{B_{\delta/(2\varepsilon)}} |\Delta z|\big|z'(r)\big|\mathrm{O}\big(r^2\big)\,\mathrm{d}x$$

$$= -\omega_{n-1}(n-4)^2 \frac{\mathrm{Scal}_g(x_0)}{3n}\varepsilon^2 \int_0^{\delta/(2\varepsilon)} \frac{6+2s^2}{(1+s^2)^5} s^7 \,\mathrm{d}s + \mathrm{O}\big(\varepsilon^2\big)$$

$$= \mathrm{O}\big(\varepsilon^2\big),$$

and

$$\int_{B_{\delta/2}} \big|z_\varepsilon'(r)\partial_r\big(\ln\sqrt{\det g}\,\big)\big|^2 \,\mathrm{d}v_g = \varepsilon^4 \int_{B_{\delta/(2\varepsilon)}} \big|z'(r)\big|^2 \mathrm{O}\big(r^2\big)\,\mathrm{d}x = \mathrm{O}\big(\varepsilon^2\big).$$

Finally, we compute the second term of the right-hand side of (27). For $n \geqslant 7$, we have:

$$\int_{B_\delta\setminus B_{\delta/2}} \big|\eta\Delta_g z_\varepsilon + 2\langle\nabla_g\eta, \nabla_g z_\varepsilon\rangle + (\Delta_g\eta) z_\varepsilon\big|^2 \,\mathrm{d}v_g$$

$$= \mathrm{O}(1)\Bigg[\int_{B_\delta\setminus B_{\delta/2}} |\Delta_g z_\varepsilon|^2 \,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} |\nabla_g z_\varepsilon|^2 \,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} z_\varepsilon^2 \,\mathrm{d}x\Bigg]$$

$$= \mathrm{O}(1)\Bigg[\int_{B_\delta\setminus B_{\delta/2}} |\Delta z_\varepsilon|^2 \,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} \big|z_\varepsilon'(r)\big|^2 r^2 \,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} \big|z_\varepsilon'(r)\big|^2 \,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} z_\varepsilon^2 \,\mathrm{d}x\Bigg]$$

$$= \mathrm{O}(1)\Bigg[\int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} |\Delta z|^2\,\mathrm{d}x + \varepsilon^4 \int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} \left|z'(r)\right|^2 r^2\,\mathrm{d}x + \varepsilon^2 \int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} \left|z'(r)\right|^2\,\mathrm{d}x$$

$$+ \varepsilon^4 \int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} z^2\,\mathrm{d}x\Bigg]$$

$$= \mathrm{o}\big(\varepsilon^2\big),$$

while for $n = 6$ we get:

$$\int\limits_{B_\delta\setminus B_{\delta/2}} \left|\eta\Delta_g z_\varepsilon + 2\langle\nabla_g\eta, \nabla_g z_\varepsilon\rangle + (\Delta_g\eta)z_\varepsilon\right|^2\,\mathrm{d}v_g = \delta^2\mathrm{O}\big(\varepsilon^2|\ln\varepsilon|\big).$$

Thus, we conclude for $n \geqslant 7$ that

$$\|\Delta_g u_\varepsilon\|^2_{L^2(M)} = \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - \frac{\mathrm{Scal}_g(x_0)}{6n} I_4\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big) \tag{28}$$

and for $n = 6$, choosing δ small enough, that

$$\|\Delta_g u_\varepsilon\|^2_{L^2(M)} = \|\Delta z\|^2_{L^2(\mathbb{R}^n)} - C(\varepsilon)\,\mathrm{Scal}_g(x_0)\varepsilon^2|\ln\varepsilon|, \tag{29}$$

where $C(\varepsilon)$ approaches a positive number as $\varepsilon \to 0$.

(4) Conclusion.

From (23) and estimates (25), (26) and (29), we obtain for $n = 6$,

$$\frac{\|u_\varepsilon\|^2_{L^{2^*}(M)} - K^2\|\Delta_g u_\varepsilon\|^2_{L^2(M)}}{\|u_\varepsilon\|^2_{L^2(M)}} = \frac{C(\varepsilon)\,\mathrm{Scal}_g(x_0)\varepsilon^2|\ln\varepsilon| + \mathrm{O}(\varepsilon^2)}{\mathrm{O}(\varepsilon^2)} \to +\infty$$

as $\varepsilon \to 0$.

If $n \geqslant 7$, from (23), (25), (26) and (28), we have:

$$\frac{\|u_\varepsilon\|^2_{L^{2^*}(M)} - K^2\|\Delta_g u_\varepsilon\|^2_{L^2(M)}}{\|u_\varepsilon\|^2_{L^2(M)}} = \frac{K^2\frac{\mathrm{Scal}_g(x_0)}{6n}(I_4 - \frac{2}{2^*}\frac{I_2I_3}{I_1})\varepsilon^2 + \mathrm{o}(\varepsilon^2)}{\mathrm{O}(\varepsilon^3)} \to +\infty$$

as $\varepsilon \to 0$, if and only if

$$\frac{2^*}{2}\frac{I_1I_4}{I_2I_3} > 1.$$

By direct computation, we find:

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^n} z^{2^*}\,\mathrm{d}x = \omega_{n-1}\int_0^\infty \frac{r^{n-1}}{(1+r^2)^n}\,\mathrm{d}r = \frac{\omega_n}{2^n},\\
I_2 &= \int_{\mathbb{R}^n} z^{2^*}|x|^2\,\mathrm{d}x = \omega_{n-1}\int_0^\infty \frac{r^{n+1}}{(1+r^2)^n}\,\mathrm{d}r = \frac{\omega_n n}{2^n(n-2)},\\
I_3 &= \int_{\mathbb{R}^n} |\Delta z|^2\,\mathrm{d}x = \omega_{n-1}(n-4)^2\int_0^\infty \frac{(n+2r^2)^2}{(1+r^2)^n} r^{n-1}\,\mathrm{d}r = \frac{\omega_n n(n-4)(n^2-4)}{2^n},\\
I_4^1 &= \int_{\mathbb{R}^n} |\Delta z|^2 r^2\,\mathrm{d}x = \omega_{n-1}(n-4)^2\int_0^\infty \frac{(n+2r^2)^2}{(1+r^2)^n} r^{n+1}\,\mathrm{d}r = \frac{\omega_n n(n-4)^2(n^2+4)}{2^n(n-6)},\\
I_4^2 &= \int_{\mathbb{R}^n} |\Delta z||z'|r\,\mathrm{d}x = \omega_{n-1}(n-4)^2\int_0^\infty \frac{n+2r^2}{(1+r^2)^{n-1}} r^{n+1}\,\mathrm{d}r\\
&= \frac{\omega_n n(n-1)(n-2)(n-4)}{2^{n-1}(n-6)}.
\end{aligned}$$

Hence,

$$I_4 = I_4^1 + 4I_4^2 = \frac{\omega_n n^2(n-4)(n^2+4n-20)}{2^n(n-6)}.$$

Therefore,

$$\frac{2^*}{2}\frac{I_1 I_4}{I_2 I_3} = \frac{n(n^2+4n-20)}{(n-4)(n-6)(n+2)} > 1 \tag{30}$$

for $n \geqslant 7$, as wished. □

4. Fourth-order problems on compact manifolds

4.1. A concentration-compactness principle

As a consequence of the asymptotically sharp Sobolev inequality of Theorem 1, we obtain the following version of the concentration-compactness principle which will be used in the proof of the existence part of Theorems 3A–3C and 4A–4C.

Lemma 2 (Concentration-compactness principle). *Let (M,g) be a smooth compact Riemannian manifold, with or without boundary, of dimension $n \geqslant 3$ and $1 < p < n/2$.*

Denote $p_1^* = np/(n-p)$ and let $K_1 = K_1(n,p)$ be the best constant in the first-order Sobolev inequality, i.e.,

$$\frac{1}{K_1(n,p)} = \inf_{u \in D^{1,p}(\mathbb{R}^n)\setminus\{0\}} \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p_1^*}(\mathbb{R}^n)}}.$$

Assume that $u_m \rightharpoonup u$ in E_i and

$$|\Delta_g u_m|^p \,\mathrm{d}v_g \rightharpoonup \mu, \qquad |u_m|^{p^*} \,\mathrm{d}v_g \rightharpoonup \nu, \qquad |\nabla_g u_m|^{p_1^*} \,\mathrm{d}v_g \rightharpoonup \pi,$$

where μ, ν, π are bounded nonnegative measures. Then, there exist at most a countable set $\mathcal{J}$, $\{x_j\}_{j\in\mathcal{J}} \subset M$ and positive numbers $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}}, \{\pi_j\}_{j\in\mathcal{J}}$ such that

$$\mu \geqslant |\Delta_g u|^p \,\mathrm{d}v_g + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j}, \qquad \nu = |u|^{p_2^*} \,\mathrm{d}v_g + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j},$$
$$\pi = |\nabla_g u|^{p_1^*} \,\mathrm{d}v_g + \sum_{j\in\mathcal{J}} \pi_j \delta_{x_j},$$

with

$$\nu_j^{1/p^*} \leqslant K \mu_j^{1/p}, \qquad \pi_j^{1/p_1^*} \leqslant C K_1 \mu_j^{1/p},$$

where C is a positive constant depending only on (M,g).

Proof. Set $v_m = u_m - u$, so that $v_m \rightharpoonup 0$ in E_i. Define:

$$\omega = \nu - |u|^{p^*} \,\mathrm{d}v_g, \qquad \theta = \pi - |u|^{p_1^*} \,\mathrm{d}v_g.$$

By the Brezis–Lieb lemma,

$$|v_m|^{p^*} \,\mathrm{d}v_g \rightharpoonup \omega, \qquad |\nabla_g v_m|^{p_1^*} \,\mathrm{d}v_g \rightharpoonup \theta.$$

Up to a subsequence we can assume that

$$|\Delta_g v_m|^p \,\mathrm{d}v_g \rightharpoonup \lambda$$

for some bounded nonnegative measure λ. We have only to show that there hold reverse Hölder inequalities for each of the measures ω and θ with respect to λ. The rest of the proof is standard.

By Theorem 1, for each $\varepsilon_1 > 0$ there exists $B_{\varepsilon_1} > 0$ such that

$$\|w\|_{L^{p^*}(M)}^p \leqslant \left(K^p + \varepsilon_1\right) \|\Delta_g w\|_{L^p(M)}^p + B_{\varepsilon_1} \|w\|_{L^p(M)}^p$$

for every $w \in E_i$. Given $\varepsilon > 0$, choosing ε_1 small enough, it follows that for any $\xi \in C^\infty(M)$ we have

$$
\begin{aligned}
&\left(\int_M |\xi|^{p^*}|v_m|^{p^*}\,\mathrm{d}v_g\right)^{p/p^*}\\
&\quad\leqslant \left(K^p+\varepsilon_1\right)\left\|\Delta_g(\xi v_m)\right\|^p_{L^p(M)}+B_{\varepsilon_1}\|\xi v_m\|^p_{L^p(M)}\\
&\quad\leqslant \left(K^p+\varepsilon_1\right)(1+\varepsilon_1)\|\xi\Delta_g v_m\|^p_{L^p(M)}+C_{\varepsilon_1}\left\|\langle\nabla_g\xi,\nabla_g v_m\rangle\right\|^p_{L^p(M)}\\
&\qquad+C_{\varepsilon_1}\left\|(\Delta_g\xi)v_m\right\|^p_{L^p(M)}+B_{\varepsilon_1}\|\xi v_m\|^p_{L^p(M)}\\
&\quad\leqslant \left(K^p+\varepsilon\right)\int_M |\xi|^p|\Delta_g v_m|^p\,\mathrm{d}v_g+C_\varepsilon\max_M|\nabla_g\xi|^p\|\nabla_g v_m\|^p_{L^p(M)}\\
&\qquad+C_\varepsilon\max_M\left(|\Delta_g\xi|^p+|\xi|^p\right)\|v_m\|^p_{L^p(M)}.
\end{aligned}
$$

Since, up to a subsequence $v_m\to 0$ and $\nabla_g v_m\to 0$ in $L^p(M)$, taking the limit when $m\to\infty$ we find

$$
\left(\int_M |\xi|^{p^*}\,\mathrm{d}\omega\right)^{p/p^*}\leqslant\left(K^p+\varepsilon\right)\int_M|\xi|^p\,\mathrm{d}\lambda
$$

for all $\varepsilon>0$. Making $\varepsilon\to 0$, we obtain the first reverse Hölder inequality:

$$
\left(\int_M |\xi|^{p^*}\,\mathrm{d}\omega\right)^{1/p^*}\leqslant K\left(\int_M|\xi|^p d\lambda\right)^{1/p} \tag{31}
$$

for all $\xi\in C^\infty(M)$.

Similarly, it is well known that for each $\varepsilon_1>0$ there exists $B_{\varepsilon_1}=B(M,g,\varepsilon_1)>0$ such that

$$
\begin{aligned}
\|\nabla_g w\|^p_{L^{p_1^*}(M)}&\leqslant\left(K_1^p+\varepsilon_1\right)\left\|\nabla_g|\nabla_g w|\right\|^p_{L^p(M)}+B_{\varepsilon_1}\|\nabla_g w\|^p_{L^p(M)}\\
&\leqslant\left(K_1^p+\varepsilon_1\right)\left\|\nabla_g^2 w\right\|^p_{L^p(M)}+B_{\varepsilon_1}\|\nabla_g w\|^p_{L^p(M)}
\end{aligned}
$$

for all $w\in E_1$ or $w\in E_2$, while, according to [11], we have

$$
\begin{aligned}
\|\nabla_g w\|^p_{L^{p_1^*}(M)}&\leqslant\left(2^{p/n}lK_1^p+\varepsilon_1\right)\left\|\nabla_g|\nabla_g w|\right\|^p_{L^p(M)}+B_{\varepsilon_1}\|\nabla_g w\|^p_{L^p(M)}\\
&\leqslant\left(2^{p/n}K_1^p+\varepsilon_1\right)\left\|\nabla_g^2 w\right\|^p_{L^p(M)}+B_{\varepsilon_1}\|\nabla_g w\|^p_{L^p(M)}
\end{aligned}
$$

for all $w\in E_3$. On the other hand, according to Appendix A, there exists a positive constant $\widetilde{C}=\widetilde{C}(M,g)$ such that

$$
\left\|\nabla_g^2 w\right\|^p_{L^p(M)}\leqslant\widetilde{C}^p\left(\|\Delta_g w\|^p_{L^p(M)}+\|\nabla_g w\|^p_{L^p(M)}+\|w\|^p_{L^p(M)}\right)
$$

for all $w \in E_i$. Therefore, given $\varepsilon > 0$, with a convenient choice of ε_1, these inequalities imply

$$\|\nabla_g w\|^p_{L^{p_1^*}(M)} \leqslant \left(C^p K_1^p + \varepsilon\right)\|\Delta_g w\|^p_{L^p(M)} + C_\varepsilon \|\nabla_g w\|^p_{L^p(M)} + C_\varepsilon \|w\|^p_{L^p(M)}$$

for all $w \in E_i$. Proceeding as previously, for any $\xi \in C^\infty(M)$ we get:

$$\left(\int_M |\xi|^{p_1^*} |\nabla_g v_m|^{p_1^*} \, \mathrm{d}v_g\right)^{p/p_1^*} \leqslant \left(C^p K_1^p + \varepsilon\right) \int_M |\xi|^p |\Delta_g v_m|^p \, \mathrm{d}v_g + C_{\varepsilon,\xi} \|\nabla_g v_m\|^p_{L^p(M)} + C_{\varepsilon,\xi} \|v_m\|^p_{L^p(M)}.$$

Again, taking the limit when $m \to \infty$ and then making $\varepsilon \to 0$, we find the second reverse Hölder inequality:

$$\left(\int_M |\xi|^{p_1^*} \, \mathrm{d}\theta\right)^{1/p_1^*} \leqslant C K_1 \left(\int_M |\xi|^p \, \mathrm{d}\lambda\right)^{1/p} \tag{32}$$

for all $\xi \in C^\infty(M)$. $\square$

4.2. Proof of Theorems 3A–3C

The proof of these theorems is done through a minimization argument involving Ekeland's principle and the above version of the concentration-compactness principle (a similar idea was used recently in [6]). In order to facilitate the reading, we will often omit the element of volume $\mathrm{d}v_g$ in the notation of integrals.

The set V_i defined in the introduction is the closed differentiable manifold $V_i = F^{-1}(1)$, where $F : E_i \to \mathbb{R}$ is the continuously differentiable functional

$$F(u) = \int_M f(x)|u|^{p^*} \, \mathrm{d}v_g.$$

Thus, by Ekeland's variational principle, there exists a minimizing sequence (u_m) for J on V_i such that $\|J'(u_m)\|_{(T_{u_m} V_i)^*} \to 0$. Since J is coercive on E_i, (u_m) is bounded. Thus, up to a subsequence, we may assume that $u_m \rightharpoonup u$ in E_i, $u_m \to u$ in $H^{1,p}(M)$ and that the conclusion of the concentration-compactness principle (Lemma 2) holds. Fix $k \in \mathcal{J}$ and choose a cutoff function $\varphi_\varepsilon \in C_0^\infty(B_{2\varepsilon}(x_k))$ satisfying $0 \leqslant \varphi_\varepsilon \leqslant 1$, $\varphi_\varepsilon \equiv 1$ in $B_\varepsilon(x_k)$ and

$$|\nabla_g \varphi_\varepsilon| \leqslant \frac{C}{\varepsilon}, \qquad |\Delta_g \varphi_\varepsilon| \leqslant \frac{C}{\varepsilon^2},$$

for some constant $C > 0$ independent of ε. Write:

$$\varphi_\varepsilon u_m = \zeta_m + \left(\int_M f(x)|u_m|^{p^*}\varphi_\varepsilon \,\mathrm{d}v_g\right)u_m,$$

where

$$\zeta_m := \left[\varphi_\varepsilon - \left(\int_M f(x)|u_m|^{p^*}\varphi_\varepsilon \,\mathrm{d}v_g\right)\right]u_m \in T_{u_m}V_i.$$

Since (ζ_m) is a bounded sequence in E_i, it follows that

$$\int_M |\Delta_g u_m|^{p-2}\Delta_g u_m \Delta_g \zeta_m + \int_M a(x)|\nabla_g u_m|^{p-2}\langle\nabla_g u_m, \nabla_g \zeta_m\rangle$$
$$+ \int_M b(x)|u_m|^{p-2}u_m\zeta_m \to 0,$$

and so

$$\lim_{m\to\infty}\left(\int_M |\Delta_g u_m|^{p-2}\Delta_g u_m \Delta_g(\varphi_\varepsilon u_m) + \int_M a(x)|\nabla_g u_m|^{p-2}\bigl\langle\nabla_g u_m, \nabla_g(\varphi_\varepsilon u_m)\bigr\rangle\right.$$
$$\left. + \int_M b(x)|u_m|^{p-2}u_m(\varphi_\varepsilon u_m)\right)$$
$$= \lim_{m\to\infty}\left(\int_M f(x)|u_m|^{p^*}\varphi_\varepsilon\right)\left(\int_M |\Delta_g u_m|^p + \int_M a(x)|\nabla_g u_m|^p + \int_M b(x)|u_m|^p\right)$$
$$= \left(\int_M f(x)\varphi_\varepsilon \,\mathrm{d}\nu\right)\inf_{V_i} J.$$

On the other hand, we can also write:

$$\lim_{m\to\infty}\left(\int_M |\Delta_g u_m|^{p-2}\Delta_g u_m \Delta_g(\varphi_\varepsilon u_m) + \int_M a(x)|\nabla_g u_m|^{p-2}\bigl\langle\nabla_g u_m, \nabla_g(\varphi_\varepsilon u_m)\bigr\rangle\right.$$
$$\left. + \int_M b(x)|u_m|^{p-2}u_m(\varphi_\varepsilon u_m)\right)$$
$$= \lim_{m\to\infty}\left(\int_M \varphi_\varepsilon|\Delta_g u_m|^p + 2\int_M |\Delta_g u_m|^{p-2}\Delta_g u_m\langle\nabla_g u_m, \nabla_g\varphi_\varepsilon\rangle\right.$$

$$
\begin{aligned}
&\quad + \int_M (\Delta_g \varphi_\varepsilon) u_m |\Delta_g u_m|^{p-2} \Delta_g u_m + \int_M a(x) \varphi_\varepsilon |\nabla_g u_m|^p \\
&\quad + \int_M a(x) u_m |\nabla_g u_m|^{p-2} \langle \nabla_g \varphi_\varepsilon, \nabla_g u_m \rangle + \int_M b(x) \varphi_\varepsilon |u_m|^p \Bigg) \\
&= \int_M \varphi_\varepsilon \, \mathrm{d}\mu + \lim_{m\to\infty} \Bigg(2 \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \langle \nabla_g \varphi_\varepsilon, \nabla_g u_m \rangle \\
&\qquad + \int_M (\Delta_g \varphi_\varepsilon) u_m |\Delta_g u_m|^{p-2} \Delta_g u_m \Bigg) \\
&\quad + \int_M a(x) \varphi_\varepsilon |\nabla_g u|^p + \int_M a(x) u |\nabla_g u|^{p-2} \langle \nabla_g u, \nabla_g \varphi_\varepsilon \rangle + \int_M b(x) \varphi_\varepsilon |u|^p .
\end{aligned}
$$

We claim that

$$
\begin{aligned}
\limsup_{m\to\infty} \left| \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \langle \nabla_g \varphi_\varepsilon, \nabla_g u_m \rangle \right| &\to 0, \\
\limsup_{m\to\infty} \left| \int_M (\Delta_g \varphi_\varepsilon) u_m |\Delta_g u_m|^{p-2} \Delta_g u_m \right| &\to 0, \\
\int_M a(x) u |\nabla_g u|^{p-2} \langle \nabla_g u, \nabla_g \varphi_\varepsilon \rangle &\to 0
\end{aligned} \tag{33}
$$

as $\varepsilon \to 0$. This will follow from Hölder's inequality and another application of the concentration-compactness principle. Indeed, as $\varepsilon \to 0$,

$$
\begin{aligned}
&\limsup_{m\to\infty} \left| \int_M |\Delta_g u_m|^{p-2} \Delta_g u_m \langle \nabla_g \varphi_\varepsilon, \nabla_g u_m \rangle \right| \\
&\quad \leqslant \limsup_{m\to\infty} \Bigg[\left(\int_M |\Delta_g u_m|^p \right)^{(p-1)/p} \left(\int_{B_{2\varepsilon}(x_k) \setminus B_\varepsilon(x_k)} |\nabla_g \varphi_\varepsilon|^n \right)^{1/n} \\
&\qquad \times \left(\int_{B_{2\varepsilon}(x_k) \setminus B_\varepsilon(x_k)} |\nabla_g u_m|^{p_1^*} \right)^{1/p_1^*} \Bigg] \\
&\quad \leqslant C \left[\frac{1}{\varepsilon^n} \operatorname{vol}_g \big(B_{2\varepsilon}(x_k) \setminus B_\varepsilon(x_k) \big) \right]^{1/n} \lim_{m\to\infty} \left(\int_{B_{2\varepsilon}(x_k) \setminus B_\varepsilon(x_k)} |\nabla_g u_m|^{p_1^*} \right)^{1/p_1^*}
\end{aligned}
$$

$$\leqslant C\biggl(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |\nabla_g u|^{p_1^*} + \sum_{j\in\mathcal{J}} \pi_j \delta_{x_j}\bigl(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)\bigr)\biggr)^{1/p_1^*} \to 0,$$

$$\limsup_{m\to\infty}\biggl|\int\limits_M |\Delta_g u_m|^{p-2}\Delta_g u_m\,(\Delta_g\varphi_\varepsilon)u_m\biggr|$$

$$\leqslant \limsup_{m\to\infty}\biggl[\biggl(\int\limits_M |\Delta_g u_m|^p\biggr)^{(p-1)/p}\biggl(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |\Delta_g\varphi_\varepsilon|^{n/2}\biggr)^{2/n} \times \biggl(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |u_m|^{p^*}\biggr)^{1/p^*}\biggr]$$

$$\leqslant C\biggl[\frac{1}{\varepsilon^n}\operatorname{vol}_g\bigl(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)\bigr)\biggr]^{2/n}\lim_{m\to\infty}\biggl(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |u_m|^{p^*}\biggr)^{1/p^*}$$

$$\leqslant C\biggl(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |u|^{p^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j}\bigl(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)\bigr)\biggr)^{1/p^*} \to 0,$$

and

$$\int\limits_M a(x)u|\nabla_g u|^{p-2}\langle\nabla_g u, \nabla_g\varphi_\varepsilon\rangle$$

$$\leqslant \max_M |a|\biggl[\biggl(\int\limits_M |u|^{p^*}\biggr)^{1/p^*}\biggl(\int\limits_M |\nabla_g u|^{p_1^*}\biggr)^{(p-1)/p_1^*} \times \biggl(\int\limits_{B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)} |\nabla_g\varphi_\varepsilon|^{n/(p+1)}\biggr)^{(p+1)/n}\biggr]$$

$$\leqslant C\biggl[\frac{1}{\varepsilon^{n/(p+1)}}\operatorname{vol}_g\bigl(B_{2\varepsilon}(x_k)\setminus B_\varepsilon(x_k)\bigr)\biggr]^{(p+1)/n}$$

$$\leqslant C\bigl(\varepsilon^{n-n/(p+1)}\bigr)^{(p+1)/n} = C\varepsilon^p \to 0.$$

Therefore, making $\varepsilon \to 0$ and using (33), we conclude that

$$\mu_k = f(x_k)\nu_k \inf_{V_i} J.$$

From the coercivity of J, we know that $\inf_{V_i} J > 0$, whence we conclude that $f(x_k) > 0$. Using the concentration-compactness principle, we obtain:

$$\mu_k \geqslant \frac{1}{K^{n/2}(f(x_k)\inf_{V_i} J)^{n/2p^*}}.$$

In particular, since μ is a bounded measure, $\mathcal{J}$ is a finite set. We assert that $\mathcal{J} = \emptyset$. On the contrary, if there exists some $k \in \mathcal{J}$, then, using the concentration-compactness principle and the coercivity of the functional J, we obtain:

$$\begin{aligned}
\inf_{V_i} J &= \lim_{m\to\infty}\biggl(\int_M |\Delta_g u_m|^p \,\mathrm{d}v_g + \int_M a(x)|\nabla_g u_m|^p \,\mathrm{d}v_g + \int_M b(x)|u_m|^p \,\mathrm{d}v_g\biggr) \\
&\geqslant \int_M |\Delta_g u|^p \,\mathrm{d}v_g + \int_M a(x)|\nabla_g u|^p \,\mathrm{d}v_g + \int_M b(x)|u|^p \,\mathrm{d}v_g + \sum \mu_j \\
&\geqslant \mu_k \geqslant \frac{1}{K^{n/2}(f(x_k)\inf_{V_i} J)^{n/2p^*}} \geqslant \frac{1}{K^{n/2}(\max_M f)^{n/2p^*}(\inf_{V_i} J)^{n/2p^*}},
\end{aligned}$$

which implies

$$\inf_{V_i} J \geqslant \frac{1}{K^p(\max_M f)^{p/p^*}},$$

contradicting (H$_i$). Brezis–Lieb lemma then implies $u_m \to u$ in $L^{p^*}(M)$, whence $\int_M f(x)|u|^{p^*}\,\mathrm{d}v_g = 1$, i.e., $u \in V_i$. As

$$\begin{aligned}
&\int_M |\Delta_g u|^p + \int_M a(x)|\nabla_g u|^p + \int_M b(x)|u|^p \\
&\quad\leqslant \liminf \int_M |\Delta_g u_m|^p + \int_M a(x)|\nabla_g u|^p + \int_M b(x)|u|^p \\
&\quad= \liminf \int_M |\Delta_g u_m|^p + \lim \int_M a(x)|\nabla_g u_m|^p + \lim \int_M b(x)|u_m|^p \\
&\quad= \liminf \biggl(\int_M |\Delta_g u_m|^p + \int_M a(x)|\nabla_g u_m|^p + \int_M b(x)|u_m|^p\biggr) \\
&\quad= \inf_{V_i} J,
\end{aligned}$$

we conclude that u is a minimizer for J on V_i.

The regularity part of Theorems 3A and 3B follows, respectively, from applying Lemmas 3 and 4 of the next subsection with $c(x) = f(x)|u|^{2^*-2}$, and L^p and C^γ estimates for elliptic equations. The regularity part of Theorem 3C follows from applying Lemma 5 with $c(x) = f(x)|u|^{p^*-2}$ and after some iterations of L^p and C^γ estimates to each equation of the system:

$$\begin{cases} -\Delta_g u = |v|^{(2-p)/(p-1)} v, & \\ -\Delta_g v = f(x)|u|^{p^*-2}u - b(x)|u|^{p-2}u & \text{in } M, \\ u = v = 0 & \text{on } \partial M. \end{cases}$$

It remains to show the positivity of solutions in Theorems 3A and 3C; this follows from an adaptation of the arguments of van der Vorst [33].

Assume first that $p = 2$, $f \geqslant 0$, a is a positive constant and that $b(x) \leqslant a^2/4$. Let u be a minimizing solution of (P_1) or (P_3), according to the case considered, and let v be the positive solution of the problem

$$-\Delta v + \frac{a}{2} v = \left| -\Delta u + \frac{a}{2} u \right| \quad \text{in } M,$$

satisfying $u = 0$ on ∂M, if M has boundary. It follows from the maximum principle that $v \geqslant |u|$. Squaring both sides of the above equation and then integrating over M, we obtain:

$$\int_M |\Delta v|^2 + a \int_M (-\Delta v)v + \frac{a^2}{4} \int_M v^2 = \int_M |\Delta u|^2 + a \int_M (-\Delta u)u + \frac{a^2}{4} \int_M u^2,$$

whence

$$\int_M |\Delta v|^2 + a \int_M |\nabla v|^2 + \int_M b(x) v^2 + \int_M \left(\frac{a^2}{4} - b(x) \right) \left(v^2 - u^2 \right)$$
$$= \int_M |\Delta u|^2 + a \int_M |\nabla u|^2 + \int_M b(x) u^2.$$

Since $b(x) \leqslant a^2/4$ and $f(x) \geqslant 0$, we conclude that $J(v) \leqslant J(u)$, and hence v is a positive minimizing solution to (P_1).

Now assume $1 < p < n/2$, $f \geqslant 0$, $a = 0$ and $b \leqslant 0$. Let v be a positive solution of

$$\begin{cases} -\Delta v = |-\Delta u| & \text{in } M, \\ v = 0 & \text{on } \partial M. \end{cases}$$

By the maximum principle, $v \geqslant |u|$. Raising this equation to the power p and integrating over M, we conclude that

$$J(v) = \frac{\int_M |\Delta v|^p + \int_M b(x) v^p}{\int_M f(x) v^{p^*}} \leqslant \frac{\int_M |\Delta u|^p + \int_M b(x)|u|^p}{\int_M f(x)|u|^{p^*}} = J(u). \qquad \square$$

4.3. Regularity

Lemma 3. *Let (M, g) be a smooth compact Riemannian manifold without boundary of dimension $n \geqslant 5$. Assume that $a \in C^1(M)$, $b \in C^0(M)$, $c \in L^{n/4}(M)$ and that the homogeneous equation*

$$\Delta_g^2 u - \operatorname{div}_g\big(a(x)\nabla_g u\big) + b(x)u = 0 \quad \text{in } M$$

admits in $H^{2,2}(M)$ only the trivial solution. If $u \in H^{2,2}(M)$ is a weak solution of the nonhomogeneous equation,

$$\Delta_g^2 u - \operatorname{div}_g\big(a(x)\nabla_g u\big) + b(x)u = c(x)u \quad \text{in } M, \tag{34}$$

then $u \in L^s(M)$ for all $1 \leqslant s < \infty$.

Proof. Given $k > 0$, define:

$$d_k(x) = \begin{cases} c(x) & \text{if } |c(x)| > k \text{ or } |u(x)| > k, \\ 0 & \text{if } |c(x)| \leqslant k \text{ and } |u(x)| \leqslant k, \end{cases}$$

and

$$e_k(x) = \big(c(x) - d_k(x)\big)u.$$

For each $k > 0$, we have $d_k \in L^{n/4}(M)$ and $e_k \in L^\infty(M)$. Moreover, given $\varepsilon > 0$, there exists k_ε such that $\|d_k\|_{L^{n/4}(M)} \leqslant \varepsilon$ for all $k \geqslant k_\varepsilon$. It follows from the hypothesis and standard elliptic L^p-theory that the operator $L = \Delta_g^2 - \operatorname{div}_g(a(x)\nabla_g) + b(x) : H^{4,t}(M) \to L^t(M)$ is an isomorphism for any $1 < t < \infty$. Therefore, for each $1 < s < \infty$, we may define the bounded linear operator $T_\varepsilon : L^s(M) \to H^{4,t}(M)$, where $t = ns/(n + 4s)$, by $T_\varepsilon w = L^{-1}(d_{k_\varepsilon} w)$. In particular, if $u \in H^{2,2}(M)$ is a weak solution of (34), then

$$u - T_\varepsilon u = L^{-1}(e_{k_\varepsilon}). \tag{35}$$

Using the critical Sobolev embedding $H^{4,t}(M) \hookrightarrow L^s(M)$, we may consider T_ε as an operator from $L^s(M)$ into $L^s(M)$. We assert that

$$\|T_\varepsilon\|_{\mathcal{L}(L^s(M))} \leqslant C\varepsilon \tag{36}$$

for some positive constant $C = C(s)$ and, consequently, the operator $I - T_\varepsilon$ is invertible for every ε sufficiently small. Indeed, by the Sobolev embedding and Hölder's inequality,

$$\begin{aligned} \|T_\varepsilon w\|_{L^s(M)} &\leqslant C\|T_\varepsilon w\|_{H^{4,ns/(n+4s)}(M)} \leqslant C\|d_{k_\varepsilon} w\|_{L^{ns/(n+4s)}(M)} \leqslant C\|d_{k_\varepsilon}\|_{L^{n/4}(M)}\|w\|_{L^s(M)} \\ &\leqslant C\varepsilon\|w\|_{L^s(M)}. \end{aligned}$$

Taking ε sufficiently small, it follows from (35) and (36) that

$$u = \sum_{n=0}^{\infty} T_\varepsilon^n\big(L^{-1}(e_{k_\varepsilon})\big),$$

which ends the proof of this lemma, since $L^{-1}(e_{k_\varepsilon}) \in L^s(M)$ for all $1 \leqslant s < \infty$. $\square$

The same proof applies to the next lemma.

Lemma 4. *Let (M,g) be a smooth compact Riemannian manifold with boundary of dimension $n \geqslant 5$. Assume that $a \in C^1(M)$, $b \in C^0(M)$, $c \in L^{n/4}(M)$ and that the homogeneous problem*:

$$\begin{cases} \Delta_g^2 u - \operatorname{div}_g\big(a(x)\nabla_g u\big) + b(x)u = 0 & \text{in } M,\\ u = \nabla_g u = 0 & \text{on } \partial M, \end{cases}$$

admits in $H_0^{2,2}(M)$ only the trivial solution. If $u \in H_0^{2,2}(M)$ is a weak solution of the nonhomogeneous problem:

$$\begin{cases} \Delta_g^2 u - \operatorname{div}_g\big(a(x)\nabla_g u\big) + b(x)u = c(x)u & \text{in } M,\\ u = \nabla_g u = 0 & \text{on } \partial M, \end{cases} \tag{37}$$

then $u \in L^s(M)$ for all $1 \leqslant s < \infty$.

Lemma 5. *Let (M,g) be a smooth compact Riemannian manifold with boundary of dimension $n \geqslant 3$ and $1 < p < n/2$. Assume that $b \in C^0(M)$, $c \in L^{n/(2p)}(M)$ and that either $a = 0$, or $p = 2$ and a is a nonnegative constant. If $u \in H^{2,p}(M) \cap H_0^{1,p}(M)$ is a weak solution of*

$$\begin{cases} \Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) - \operatorname{div}_g\big(a|\nabla_g u|^{p-2}\nabla_g u\big) + b(x)|u|^{p-2}u \\ \quad = c(x)|u|^{p-2}u \quad \text{in } M,\\ u = \Delta_g u = 0 \qquad \text{on } \partial M, \end{cases} \tag{38}$$

then $u \in L^s(M)$ for all $1 \leqslant s < \infty$.

Proof. Assume first $a = 0$. Denoting $c_0 = c - b \in L^{n/(2p)}(M)$, (38) takes the form:

$$\begin{cases} \Delta_g\big(|\Delta_g u|^{p-2}\Delta_g u\big) = c_0(x)|u|^{p-2}u & \text{in } M,\\ u = \Delta_g u = 0 & \text{on } \partial M. \end{cases} \tag{39}$$

In order to obtain regularity, it is convenient to write (39) as a coupled elliptic system with Dirichlet boundary condition. Define:

$$v = -|\Delta_g u|^{p-2}\Delta_g u \in L^{p/(p-1)}(M).$$

We assert that $v \in H^{2,q}(M) \cap H_0^{1,q}(M)$, with $q = np/((n+2)p - n) > 1$. Indeed, clearly

$$-\int_M v \Delta_g \varphi \, \mathrm{d}v_g = \int_M c_0(x)|u|^{p-2} u \varphi \, \mathrm{d}v_g$$

for every $\varphi \in H^{2,p}(M) \cap H_0^{1,p}(M)$. Noticing that $u \in L^{p^*}(M)$ implies $c_0(x)|u|^{p-2}u \in L^q(M)$, let $w \in H^{2,q}(M) \cap H_0^{1,q}(M)$ be a solution of the Dirichlet problem:

$$\begin{cases} -\Delta_g w = c_0(x)|u|^{p-2}u & \text{in } M, \\ w = 0 & \text{on } \partial M. \end{cases}$$

It follows that

$$\int_M (v - w) \Delta_g \varphi \, \mathrm{d}v_g = 0$$

for every $\varphi \in H^{2,p}(M) \cap H_0^{1,p}(M)$. Hence, $v = w$, proving our assertion. We can thus rewrite (39) as

$$\begin{cases} -\Delta_g u = |v|^{(2-p)/(p-1)} v, & \\ -\Delta_g v = c_0(x)|u|^{p-2}u & \text{in } M, \\ u = v = 0 & \text{on } \partial M. \end{cases} \tag{40}$$

Given $k > 0$, define:

$$d_k(x) = \begin{cases} c_0(x) & \text{if } |c_0(x)| > k \text{ or } |u(x)| > k, \\ 0 & \text{if } |c_0(x)| \leqslant k \text{ and } |u(x)| \leqslant k, \end{cases}$$

and

$$e_k(x) = \big(c_0(x) - d_k(x)\big)|u|^{p-2}u.$$

Again, we have $d_k \in L^{n/(2p)}(M)$ and $e_k \in L^\infty(M)$ for every $k > 0$. Furthermore, given $\varepsilon > 0$, there exists k_ε such that $\|d_k\|_{L^{n/(2p)}(M)} \leqslant \varepsilon$ for all $k \geqslant k_\varepsilon$.

Since $-\Delta_g : H^{2,t}(M) \cap H_0^{1,t}(M) \to L^t(M)$ is an isomorphism for each $1 < t < \infty$, given

$$\max\left\{\frac{n}{n-2}, \frac{1}{p-1}\right\} < t < \frac{n}{2(p-1)},$$

we can write

$$v - T_\varepsilon v = (-\Delta_g)^{-1}(e_{k_\varepsilon}), \tag{41}$$

where $T_\varepsilon : L^t(M) \to H^{2,nt/(n+2t)}(M) \cap H_0^{1,nt/(n+2t)}(M)$ is the homogeneous operator

$$T_\varepsilon w = (-\Delta_g)^{-1}\big(d_{k_\varepsilon}\big|(-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\big|^{p-2}(-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\big).$$

Thanks to the critical Sobolev embedding $H^{2,nt/(n+2t)}(M) \hookrightarrow L^t(M)$, we may see T_ε as an operator from $L^t(M)$ into $L^t(M)$. Considering the usual norm on the space of homogeneous operators, we claim that

$$\|T_\varepsilon\| \leqslant C\varepsilon \tag{42}$$

for some positive constant $C = C(t)$. Indeed, by the boundedness of $(-\Delta_g)^{-1}$ and Hölder's inequality, we have

$$\begin{aligned}
\|T_\varepsilon w\|_{L^t(M)} &\leqslant C\Big\|d_{k_\varepsilon}\big|(-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\big|^{p-2} \\
&\qquad \times (-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\Big\|_{L^{nt/(n+2t)}(M)} \\
&\leqslant C\|d_{k_\varepsilon}\|_{L^{n/(2p)}(M)}\Big\|\big|(-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\big|^{p-2} \\
&\qquad\qquad \times (-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\Big\|_{L^{nt/(n-2(p-1)t)}(M)} \\
&\leqslant C\varepsilon\Big\|(-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\Big\|^{p-1}_{L^{n(p-1)t/(n-2(p-1)t)}(M)} \\
&\leqslant C\varepsilon\Big\|(-\Delta_g)^{-1}\big(|w|^{(2-p)/(p-1)}w\big)\Big\|^{p-1}_{H^{2,t(p-1)}(M)} \\
&\leqslant C\varepsilon\Big\||w|^{(2-p)/(p-1)}w\Big\|^{p-1}_{L^{t(p-1)}(M)} \\
&\leqslant C\varepsilon\|w\|_{L^t(M)},
\end{aligned}$$

which proves the assertion.

Choose $t = p/(p-1)$. Noticing that $v \in L^t(M)$ and the space of homogeneous operators under the standard norm is Banach, it follows from (41) and (42) that

$$v = \sum_{n=0}^{\infty} T_\varepsilon^n\big(L^{-1}(e_{k_\varepsilon})\big)$$

if ε is sufficiently small. This implies that $v \in L^t(M)$ for every $\max\{n/(n-2), 1/(p-1)\} < t < n/(2(p-1))$. Let

$$t = \frac{ns}{(p-1)(n+2s)}$$

with $s > \max\{(p-1)n/(n-2p), n/(n-2)\}$. Clearly t is in the admissible range. Then, from the critical Sobolev embedding $H^{2,t(p-1)}(M) \hookrightarrow L^s(M)$, it follows that

$$\begin{aligned}\|u\|_{L^s(M)} &= \left\|(-\Delta_g)^{-1}\left(|v|^{(2-p)/(p-1)}v\right)\right\|_{L^s(M)}\\ &\leqslant C\left\|(-\Delta_g)^{-1}\left(|v|^{(2-p)/(p-1)}v\right)\right\|_{H^{2,t(p-1)}(M)}\\ &\leqslant C\left\||v|^{(2-p)/(p-1)}v\right\|_{L^{t(p-1)}(M)} = C\|v\|_{L^t(M)}^{1/(p-1)}\end{aligned}$$

and hence we conclude that $u \in L^s(M)$ for all $s > \max\{(p-1)n/(n-2p), n/(n-2)\}$. This finishes the proof in the case $a = 0$.

If $p = 2$ and a is a nonnegative constant, we consider instead the system:

$$\begin{cases} -\Delta_g u = v, & \\ -\Delta_g v + av = c_0(x)u & \text{in } M,\\ u = v = 0 & \text{on } \partial M,\end{cases}$$

and the proof is analogous. □

4.4. *Proof of Corollary 2*

Proceeding as in the proof of Theorem 2, consider a geodesic ball $B_\delta(x_0) \subset (M\backslash\partial M)$, a radial cutoff function $\eta \in C_0^\infty(B_\delta)$ satisfying $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in $B_{\delta/2}$ and $\eta = 0$ in $\mathbb{R}^n\backslash B_\delta$, and define, up to the exponential chart $\exp_{x_0}$,

$$u_\varepsilon(x) = \eta(x) z_\varepsilon(x),$$

where

$$z_\varepsilon(x) = \varepsilon^{-n/p^*} z\left(\frac{x}{\varepsilon}\right)$$

with z being a positive radial minimizer for the Sobolev quotient (4). By Theorems 3A, 3B or 3C, according to which case we are dealing with, it is enough to show that for some sufficiently small ε we have

$$\frac{\int_M |\Delta_g u_\varepsilon|^p \, \mathrm{d}v_g + \int_M b(x)|u_\varepsilon|^p \, \mathrm{d}v_g}{(\int_M f(x)|u_\varepsilon|^{p^*} \, \mathrm{d}v_g)^{p/p^*}} < \frac{1}{K^p f(x_0)^{p/p^*}}.$$

Considering the expansions $\eta(x) = 1 + \mathrm{O}(r^3)$, (24) and

$$f(x) = f(x_0) + \frac{1}{2}\sum_{i,j=1}^n \partial_{ij} f(x_0) x_i x_j + \mathrm{O}\left(r^3\right),$$

and noticing that $\Delta_g f(x_0) = \sum_{i=1}^n \partial_{ii} f(x_0)$, we can write:

$$
\begin{aligned}
&\int_M f(x)|u_\varepsilon|^{p^*}\,\mathrm{d}v_g \\
&\quad = f(x_0)\int_M |u_\varepsilon|^{p^*}\,\mathrm{d}v_g + \frac{1}{2}\sum_{i,j=1}^{n}\partial_{ij}f(x_0)\int_M |u_\varepsilon|^{p^*}x_ix_j\,\mathrm{d}v_g + \int_M |u_\varepsilon|^{p^*}\mathrm{O}\big(r^3\big)\,\mathrm{d}v_g \\
&\quad = f(x_0)\int_{B_\delta} |u_\varepsilon|^{p^*}\,\mathrm{d}x + \frac{3\Delta_g f(x_0) - f(x_0)\,\mathrm{Scal}_g(x_0)}{6n}\int_{B_\delta}|u_\varepsilon|^{p^*}r^2\,\mathrm{d}x \\
&\qquad + \int_{B_\delta}|u_\varepsilon|^{p^*}\mathrm{O}\big(r^3\big)\,\mathrm{d}x \\
&\quad = f(x_0)\int_{\mathbb{R}^n} z^{p^*}\,\mathrm{d}x - f(x_0)\int_{\mathbb{R}^n\setminus B_{\delta/\varepsilon}} z^{p^*}\,\mathrm{d}x + \frac{3\Delta_g f(x_0) - f(x_0)\,\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2\int_{\mathbb{R}^n} z^{p^*}r^2\,\mathrm{d}x \\
&\qquad - \frac{3\Delta_g f(x_0) - f(x_0)\,\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2\int_{\mathbb{R}^n\setminus B_{\delta/\varepsilon}} z^{p^*}r^2\,\mathrm{d}x + \varepsilon^3\int_{B_{\delta/\varepsilon}} z^{p^*}\mathrm{O}\big(r^3\big)\,\mathrm{d}x.
\end{aligned}
$$

By straightforward computations from the asymptotic behavior of z provided in Appendix B, we obtain for $n(n+2)/(n^2+4) < p < n/2$,

$$
\begin{aligned}
\int_M f(x)|u_\varepsilon|^{p^*}\,\mathrm{d}v_g &= f(x_0)\int_{\mathbb{R}^n} z^{p^*}\,\mathrm{d}x + I_2\frac{3\Delta_g f(x_0) - f(x_0)\,\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big) \\
&= f(x_0)\bigg(\int_{\mathbb{R}^n} z^{p^*}\,\mathrm{d}x\bigg)\bigg[1 + \frac{1}{n}\frac{I_2}{I_1}\bigg(\frac{\Delta_g f(x_0)}{2f(x_0)} - \frac{\mathrm{Scal}_g(x_0)}{6}\bigg)\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big)\bigg].
\end{aligned}
$$

Thus, we get:

$$
\begin{aligned}
&\bigg(\int_M f(x)|u_\varepsilon|^{p^*}\,\mathrm{d}v_g\bigg)^{p/p^*} \\
&\quad = f(x_0)^{p/p^*}\|z\|^p_{L^{p^*}(\mathbb{R}^n)}\bigg[1 + \frac{p}{np^*}\frac{I_2}{I_1}\bigg(\frac{\Delta_g f(x_0)}{2f(x_0)} - \frac{\mathrm{Scal}_g(x_0)}{6}\bigg)\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big)\bigg]. \qquad (43)
\end{aligned}
$$

For the next estimate, we write:

$$
\begin{aligned}
\int_M |\Delta_g u_\varepsilon|^p\,\mathrm{d}v_g &= \int_{B_\delta}\big|\eta\Delta_g z_\varepsilon + 2\langle\nabla_g\eta, \nabla_g z_\varepsilon\rangle + (\Delta_g\eta)z_\varepsilon\big|^p\,\mathrm{d}v_g \\
&= \int_{B_{\delta/2}} |\Delta_g z_\varepsilon|^p\,\mathrm{d}v_g
\end{aligned}
$$

$$+\int_{B_\delta\setminus B_{\delta/2}} \left|\eta\Delta_g z_\varepsilon + 2\langle\nabla_g\eta,\nabla_g z_\varepsilon\rangle + (\Delta_g\eta)z_\varepsilon\right|^p \mathrm{d}v_g. \tag{44}$$

The estimate of the first term of the right-hand side of (44) requires the elementary inequality

$$|1+t|^p \leqslant 1 + pt + C_{1,p}t^2 + C_{2,p}|t|^p$$

valid for all $t\in\mathbb{R}$, where $C_{1,p}$ and $C_{2,p}$ are some large constants depending only on p, except in the case $1\leqslant p\leqslant 2$, when $C_{1,p}=0$. Writing the Laplacian in normal geodesic coordinates, since $\Delta z_\varepsilon(r)<0$ and $z_\varepsilon'(r)<0$ for all $r>0$, it follows that

$$\begin{aligned}
|\Delta_g z_\varepsilon|^p &= \left|\Delta z_\varepsilon + z_\varepsilon'(r)\partial_r\left(\ln\sqrt{\det g}\right)\right|^p = |\Delta z_\varepsilon|^p\left|1+\frac{z_\varepsilon'(r)\partial_r(\ln\sqrt{\det g})}{\Delta z_\varepsilon}\right|^p\\
&\leqslant |\Delta z_\varepsilon|^p\left(1+p\frac{z_\varepsilon'(r)\partial_r(\ln\sqrt{\det g})}{\Delta z_\varepsilon}+C_{1,p}\left|\frac{z_\varepsilon'(r)\partial_r(\ln\sqrt{\det g})}{\Delta z_\varepsilon}\right|^2\right.\\
&\qquad\left.+C_{2,p}\left|\frac{z_\varepsilon'(r)\partial_r(\ln\sqrt{\det g})}{\Delta z_\varepsilon}\right|^p\right)\\
&= |\Delta z_\varepsilon|^p + p|\Delta z_\varepsilon|^{p-1}\left|z_\varepsilon'(r)\right|\partial_r\left(\ln\sqrt{\det g}\right)\\
&\quad + C_{1,p}|\Delta z_\varepsilon|^{p-2}\left|z_\varepsilon'(r)\partial_r\left(\ln\sqrt{\det g}\right)\right|^2 + C_{2,p}\left|z_\varepsilon'(r)\partial_r\left(\ln\sqrt{\det g}\right)\right|^p.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{B_{\delta/2}} |\Delta_g z_\varepsilon|^p\,\mathrm{d}v_g &\leqslant \int_{B_{\delta/2}} |\Delta z_\varepsilon|^p\,\mathrm{d}v_g + p\int_{B_{\delta/2}} |\Delta z_\varepsilon|^{p-1}\left|z_\varepsilon'(r)\right|\partial_r\left(\ln\sqrt{\det g}\right)\mathrm{d}v_g\\
&\quad + C_{1,p}\int_{B_{\delta/2}} |\Delta z_\varepsilon|^{p-2}\left|z_\varepsilon'(r)\partial_r\left(\ln\sqrt{\det g}\right)\right|^2\mathrm{d}v_g\\
&\quad + C_{2,p}\int_{B_{\delta/2}} \left|z_\varepsilon'(r)\partial_r\left(\ln\sqrt{\det g}\right)\right|^p\mathrm{d}v_g.
\end{aligned}$$

Again using (24), it follows by straightforward computation from the asymptotic behaviors of Δz and z' given in Appendix B that

$$\begin{aligned}
\int_{B_{\delta/2}} |\Delta z_\varepsilon|^p\,\mathrm{d}v_g &= \int_{\mathbb{R}^n}|\Delta z|^p\,\mathrm{d}x - \int_{\mathbb{R}^n\setminus B_{\delta/(2\varepsilon)}} |\Delta z|^p\,\mathrm{d}x - \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2\int_{\mathbb{R}^n}|\Delta z|^p r^2\,\mathrm{d}x\\
&\quad + \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2\int_{\mathbb{R}^n\setminus B_{\delta/(2\varepsilon)}} |\Delta z|^p r^2\,\mathrm{d}x + \varepsilon^3\int_{B_{\delta/(2\varepsilon)}} |\Delta z|^p\mathrm{O}\left(r^3\right)\mathrm{d}x
\end{aligned}$$

$$= \|\Delta z\|^p_{L^p(\mathbb{R}^n)} - I_4^1 \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 + \mathrm{o}(\varepsilon^2),$$

if $1 < p < (n+2)/4$,

$$\begin{aligned}
&\int_{B_{\delta/2}} |\Delta z_\varepsilon|^{p-1} |z'_\varepsilon(r)| \partial_r \big(\ln\sqrt{\det g}\big)\,\mathrm{d}v_g \\
&= -\frac{\mathrm{Scal}_g(x_0)}{3n} \int_{B_{\delta/2}} |\Delta z_\varepsilon|^{p-1}|z'_\varepsilon(r)| r\,\mathrm{d}x + \int_{B_{\delta/2}} |\Delta z_\varepsilon|^{p-1}|z'_\varepsilon(r)|\mathrm{O}(r^2)\,\mathrm{d}x \\
&= -\frac{\mathrm{Scal}_g(x_0)}{3n}\varepsilon^2 \int_{\mathbb{R}^n} |\Delta z|^{p-1}|z'(r)| r\,\mathrm{d}x + \frac{\mathrm{Scal}_g(x_0)}{3n}\varepsilon^2 \int_{\mathbb{R}^n\setminus B_{\delta/(2\varepsilon)}} |\Delta z|^{p-1}|z'(r)| r\,\mathrm{d}x \\
&\quad + \varepsilon^3 \int_{B_{\delta/(2\varepsilon)}} |\Delta z|^{p-1}|z'(r)|\mathrm{O}(r^2)\,\mathrm{d}x \\
&= -I_4^2 \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 + \mathrm{o}(\varepsilon^2)
\end{aligned}$$

if $n \geqslant 5$ and $n(n+2)/(n^2+4) < p < (n+2)/4$,

$$\int_{B_{\delta/2}} |\Delta z_\varepsilon|^{p-2}\big|z'_\varepsilon(r)\partial_r\big(\ln\sqrt{\det g}\big)\big|^2\,\mathrm{d}v_g = \varepsilon^4 \int_{B_{\delta/(2\varepsilon)}} |\Delta z|^{p-2}|z'(r)|^2\mathrm{O}(r^2)\,\mathrm{d}x = \mathrm{o}(\varepsilon^2)$$

if $n \geqslant 5$ and $2(n-1)/n < p < (n+2)/4$ (recall that for $p \leqslant 2$ this term plays no role, since in this case $C_{1,p} = 0$), and

$$\int_{B_{\delta/2}} \big|z'_\varepsilon(r)\partial_r\big(\ln\sqrt{\det g}\big)\big|^p\,\mathrm{d}v_g = \varepsilon^{2p} \int_{B_{\delta/(2\varepsilon)}} |z'(r)|^p\mathrm{O}(r^p)\,\mathrm{d}x = \mathrm{o}(\varepsilon^2)$$

if $n \geqslant 5$ and $(n+2)/n < p < (n+2)/4$.

Finally, we compute the second term of the right-hand side of (44). For $n \geqslant 5$ and $(n+2)/n < p < (n+2)/4$ we have:

$$\begin{aligned}
&\int_{B_\delta\setminus B_{\delta/2}} \big|\eta\Delta_g z_\varepsilon + 2\langle\nabla_g\eta, \nabla_g z_\varepsilon\rangle + (\Delta_g\eta)z_\varepsilon\big|^p\,\mathrm{d}v_g \\
&= \mathrm{O}(1)\Bigg[\int_{B_\delta\setminus B_{\delta/2}} |\Delta_g z_\varepsilon|^p\,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} |\nabla_g z_\varepsilon|^p\,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} z_\varepsilon^p\,\mathrm{d}x\Bigg] \\
&= \mathrm{O}(1)\Bigg[\int_{B_\delta\setminus B_{\delta/2}} |\Delta z_\varepsilon|^p\,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} |z'_\varepsilon(r)|^p r^p\,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} |z'_\varepsilon(r)|^p\,\mathrm{d}x + \int_{B_\delta\setminus B_{\delta/2}} z_\varepsilon^p\,\mathrm{d}x\Bigg]
\end{aligned}$$

$$= \mathrm{O}(1)\Bigg[\int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} |\Delta z|^p \,\mathrm{d}x + \varepsilon^{2p} \int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} \big|z'(r)\big|^p r^p \,\mathrm{d}x + \varepsilon^p \int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} \big|z'(r)\big|^p \,\mathrm{d}x$$
$$+ \varepsilon^{2p} \int\limits_{B_{\delta/\varepsilon}\setminus B_{\delta/(2\varepsilon)}} z^p \,\mathrm{d}x\Bigg]$$
$$= \mathrm{o}\big(\varepsilon^2\big).$$

Therefore,

$$\begin{aligned}\|\Delta_g u_\varepsilon\|^p_{L^p(M)} &\leqslant \|\Delta z\|^p_{L^p(\mathbb{R}^n)} - I_4^1 \frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 - pI_4^2\frac{\mathrm{Scal}_g(x_0)}{3n}\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big)\\ &= \|\Delta z\|^p_{L^p(\mathbb{R}^n)}\left[1 - \frac{I_4}{I_3}\frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 + \mathrm{o}\big(\varepsilon^2\big)\right].\end{aligned} \tag{45}$$

Finally, considering the expansions $\eta(x) = 1 + \mathrm{O}(r^3)$, $\mathrm{d}v_g = 1 + \mathrm{O}(r^2)$, and

$$b(x) = b(x_0) + \sum_{i=1}^n \partial_i b(x_0) x_i + \mathrm{O}\big(r^2\big),$$

noticing that $\int_{B_{\delta/\varepsilon}} z^p x_i \,\mathrm{d}x = 0$, we obtain for $n \geqslant 5$ and $(n+2)/n < p < (n+2)/4$ that

$$\int\limits_M b(x)|u_\varepsilon|^p \,\mathrm{d}v_g = b(x_0)\varepsilon^{2p} \int\limits_{B_{\delta/\varepsilon}} z^p \,\mathrm{d}x + \varepsilon^{2p+2} \int\limits_{B_{\delta/\varepsilon}} z^p \mathrm{O}\big(r^2\big)\,\mathrm{d}x = \mathrm{o}\big(\varepsilon^2\big). \tag{46}$$

Putting (43), (45) and (46) together, we get

$$\begin{aligned}&\frac{\int_M |\Delta_g u_\varepsilon|^p \,\mathrm{d}v_g + \int_M b(x)|u_\varepsilon|^p \,\mathrm{d}v_g}{(\int_M f(x)|u_\varepsilon|^{p^*}\,\mathrm{d}v_g)^{p/p^*}}\\ &\quad\leqslant \frac{\|\Delta z\|^p_{L^p(\mathbb{R}^n)}}{f(x_0)^{p/p^*}\|z\|^p_{L^{p^*}(\mathbb{R}^n)}} \frac{1 - \frac{I_4}{I_3}\frac{\mathrm{Scal}_g(x_0)}{6n}\varepsilon^2 + o(\varepsilon^2)}{1 + \frac{p}{np^*}\frac{I_2}{I_1}(\frac{\Delta_g f(x_0)}{2f(x_0)} - \frac{\mathrm{Scal}_g(x_0)}{6})\varepsilon^2 + \mathrm{o}(\varepsilon^2)}\\ &\quad< \frac{1}{K^p f(x_0)^{p/p^*}}\end{aligned}$$

if

$$\frac{I_4}{I_3}\frac{\mathrm{Scal}_g(x_0)}{6n} + \frac{p}{np^*}\frac{I_2}{I_1}\left(\frac{\Delta_g f(x_0)}{2f(x_0)} - \frac{\mathrm{Scal}_g(x_0)}{6}\right) > 0.$$

Since this inequality is equivalent to (8), the proof is finished. □

5. The role of the geometry on Brezis–Nirenberg type problems

Proof of Theorems 4A–4C. Because $\lambda < \lambda_1$, the functional

$$J(u) = \int_M |\Delta_g u|^p \,\mathrm{d}v_g - \lambda \int_M |u|^p \,\mathrm{d}v_g$$

is coercive on E_i. Therefore, if $p = 2$, $n \geqslant 7$ and $\mathrm{Scal}_g(x_0) > 0$, the proof follows from Corollary 2, taking $f \equiv 1$, and from (30). In the other cases, we apply Theorems 3A–3C, according to the situation. Thus, we only need to show that $\inf_{V_i} J < 1/K^p$ or, equivalently,

$$\inf_{u \in E_i \setminus \{0\}} \frac{\int_M |\Delta_g u|^p \,\mathrm{d}v_g - \lambda \int_M |u|^p \,\mathrm{d}v_g}{(\int_M |u|^{p^*} \,\mathrm{d}v_g)^{p/p^*}} < \frac{1}{K^p}. \tag{47}$$

We obtain (47) by proving that for all sufficiently small ε there holds

$$\frac{\int_M |\Delta_g u_\varepsilon|^p \,\mathrm{d}v_g - \lambda \int_M |u_\varepsilon|^p \,\mathrm{d}v_g}{(\int_M |u_\varepsilon|^{p^*} \,\mathrm{d}v_g)^{p/p^*}} < \frac{1}{K^p}, \tag{48}$$

where $u_\varepsilon \in C_0^\infty(M)$ are the functions defined in the proof of Corollary 2: here we choose x_0 to be any interior point of positive scalar curvature if there is one, or any interior point of a flat neighborhood of the manifold, when this is the case.

If $p = 2$, $n = 6$, $\mathrm{Scal}_g(x_0) > 0$ and ε is small enough, according to the estimates we did in the proof of Theorem 2 we have:

$$\frac{\int_M |\Delta_g u_\varepsilon|^2 \,\mathrm{d}v_g - \lambda \int_M |u_\varepsilon|^2 \,\mathrm{d}v_g}{(\int_M |u_\varepsilon|^{2^*} \,\mathrm{d}v_g)^{2/2^*}} = \frac{1}{K^2} \frac{1 - \mathrm{Scal}_g(x_0)\mathrm{O}(\varepsilon^2 |\ln \varepsilon|)}{1 - \mathrm{O}(\varepsilon^2)} < \frac{1}{K^2}.$$

Now, let M be flat in some neighborhood and $n/(n-2) < p < \sqrt{n/2}$. From Appendix B, it follows that

$$\begin{aligned}
&\int_M |\Delta u_\varepsilon|^p \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} |\Delta z|^p \,\mathrm{d}x - \int_{\mathbb{R}^n \setminus B_{\delta/(2\varepsilon)}} |\Delta z|^p \,\mathrm{d}x + \int_{B_\delta \setminus B_{\delta/2}} \bigl|\eta \Delta z_\varepsilon + 2\langle \nabla \eta, \nabla z_\varepsilon \rangle + (\Delta \eta) z_\varepsilon\bigr|^p \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^n} |\Delta z|^p \,\mathrm{d}x - \int_{\mathbb{R}^n \setminus B_{\delta/(2\varepsilon)}} |\Delta z|^p \,\mathrm{d}x \\
&\qquad + \mathrm{O}(1)\Biggl(\int_{B_{\delta/\varepsilon} \setminus B_{\delta/(2\varepsilon)}} |\Delta z|^p \,\mathrm{d}x + \varepsilon^p \int_{B_{\delta/\varepsilon} \setminus B_{\delta/(2\varepsilon)}} |\nabla z|^p \,\mathrm{d}x + \varepsilon^{2p} \int_{B_{\delta/\varepsilon} \setminus B_{\delta/(2\varepsilon)}} z^p \,\mathrm{d}x \Biggr) \\
&\quad = \|\Delta z\|_{L^p(\mathbb{R}^n)}^p + \mathrm{o}\bigl(\varepsilon^{2p}\bigr),
\end{aligned}$$

$$\int_M |u_\varepsilon|^p \,\mathrm{d}x = \varepsilon^{2p} \int_{B_{\delta/\varepsilon}} z^p \,\mathrm{d}x + \varepsilon^{2p+2} \int_{B_{\delta/\varepsilon} \setminus B_{\delta/(2\varepsilon)}} z^p \mathrm{O}\big(r^2\big) \,\mathrm{d}x = \mathrm{O}\big(\varepsilon^{2p}\big),$$

and

$$\begin{aligned}\int_M |u_\varepsilon|^{p^*} \,\mathrm{d}x &= \int_{\mathbb{R}^n} z^{p^*} \,\mathrm{d}x - \int_{\mathbb{R}^n \setminus B_{\delta/\varepsilon}} z^{p^*} \,\mathrm{d}x + \varepsilon^2 \int_{B_{\delta/\varepsilon} \setminus B_{\delta/(2\varepsilon)}} z^{p^*} \mathrm{O}\big(r^2\big) \,\mathrm{d}x \\ &= \|z\|^p_{L^{p^*}(\mathbb{R}^n)} - \mathrm{o}\big(\varepsilon^{2p}\big).\end{aligned}$$

Therefore, if ε is sufficiently small and $\lambda > 0$, we have:

$$\frac{\int_M |\Delta u_\varepsilon|^p \,\mathrm{d}x - \lambda \int_M |u_\varepsilon|^p \,\mathrm{d}x}{(\int_M |u_\varepsilon|^{p^*} \,\mathrm{d}x)^{p/p^*}} \leqslant \frac{\|\Delta z\|^p_{L^p(\mathbb{R}^n)} - \lambda \mathrm{O}(\varepsilon^{2p})}{\|z\|^p_{L^{p^*}(\mathbb{R}^n)} - \mathrm{o}(\varepsilon^{2p})} = \frac{1}{K^p} \frac{1 - \lambda \mathrm{O}(\varepsilon^{2p})}{1 - \mathrm{o}(\varepsilon^{2p})} < \frac{1}{K^p}.$$

If $p = \sqrt{n/2}$, then

$$\int_M |\Delta u_\varepsilon|^p \,\mathrm{d}x = \|\Delta z\|^p_{L^p(\mathbb{R}^n)} + \mathrm{O}\big(\varepsilon^{2p}\big), \qquad \int_M |u_\varepsilon|^p \,\mathrm{d}x = \mathrm{O}\big(\varepsilon^{2p} |\ln \varepsilon|\big),$$

and

$$\int_M |u_\varepsilon|^{p^*} \,\mathrm{d}x = \|z\|^p_{L^{p^*}(\mathbb{R}^n)} - \mathrm{O}\big(\varepsilon^{2p}\big),$$

hence (48) also holds in this case.

In order to show that there are positive solutions, consider any nontrivial solution $u \in E_3$ which minimizes the quotient in (47). Let $w \in E_3$ be the positive solution to the Dirichlet problem:

$$\begin{cases} -\Delta_g w = |\Delta_g u| & \text{in } M, \\ w = 0 & \text{on } \partial M. \end{cases}$$

By the maximum principle, $w \geqslant |u|$ in M. Therefore, if $\lambda \geqslant 0$,

$$\frac{\int_M |\Delta_g w|^p \,\mathrm{d}v_g - \lambda \int_M |w|^p \,\mathrm{d}v_g}{(\int_M |w|^{p^*} \,\mathrm{d}v_g)^{p/p^*}} \leqslant \frac{\int_M |\Delta_g u|^p \,\mathrm{d}v_g - \lambda \int_M |u|^p \,\mathrm{d}v_g}{(\int_M |u|^{p^*} \,\mathrm{d}v_g)^{p/p^*}} < \frac{1}{K^p}$$

and thus w is a positive solution to the problem (BN_3).

The nonexistence of positive solutions for $\lambda \geqslant \lambda_1$ in the case (BN_1) follows immediately from direct integration and in the case (BN_3) follows from Proposition 2.12 in [27], after reformulation in terms of elliptic systems. The positivity in Theorem 4C follows immediately from Theorem 3C taking $b(x) = -\lambda$. $\square$

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Appendix A. Equivalence of norms

Let (M,g) be a smooth compact n-dimensional Riemannian manifold, with or without boundary. Choose a finite set of parametrizations $\{\phi_k\colon\ \Omega_k\to U_k\}_{1\leqslant k\leqslant N}$ such that $\{U_k\}_{1\leqslant k\leqslant N}$ is a covering of M, and let $\{\eta_k\}_{1\leqslant k\leqslant N}$ be a partition of unity subordinated to this covering. Define the following norm in $C^\infty(M)$:

$$\|u\|_{2,p}=\sum_{k=1}^N\big\|(\eta_k u)\circ\phi_k\big\|_{H^{2,p}(\Omega_k)}.$$

The change of coordinates theorem ensures that the above definition does not depend neither on the chosen parametrizations, nor on the partition of unity. Moreover, in $C^\infty(M)$, the norm $\|\cdot\|_{2,p}$ is equivalent to the norm $\|\cdot\|_{H^{2,p}(M)}$. Indeed, writing ∇_g and ∇_g^2 in local coordinates, for $u\in C^\infty(M)$, we find

$$\begin{aligned}\|u\|_{H^{2,p}(M)} &= \Big\|\sum \eta_k u\Big\|_{H^{2,p}(M)}\leqslant \sum\|\eta_k u\|_{H^{2,p}(U_k)}\\ &\leqslant C\sum\Big(\big\|\partial^2\big((\eta_k u)\circ\phi_k\big)\big\|_{L^p(\Omega_k)}+\big\|\partial\big((\eta_k u)\circ\phi_k\big)\big\|_{L^p(\Omega_k)}\\ &\qquad\quad+\big\|(\eta_k u)\circ\phi_k\big\|_{L^p(\Omega_k)}\Big)\\ &\leqslant C\|u\|_{2,p}.\end{aligned}$$

On the other hand, L^p theory applied to the elliptic operator Δ_g gives us:

$$\begin{aligned}\big\|(\eta_k u)\circ\phi_k\big\|_{H^{2,p}(\Omega_k)} &\leqslant C\Big(\big\|\Delta_g\big((\eta_k u)\circ\phi_k\big)\big\|_{L^p(\Omega_k)}+\big\|(\eta_k u)\circ\phi_k\big\|_{L^p(\Omega_k)}\Big)\\ &= C\Big(\big\|\Delta_g(\eta_k u)\big\|_{L^p(U_k)}+\|\eta_k u\|_{L^p(U_k)}\Big)\\ &\leqslant C\Big(\|\Delta_g u\|_{L^p(M)}+\|\nabla_g u\|_{L^p(M)}+\|u\|_{L^p(M)}\Big)\\ &\leqslant C\Big(\big\|\nabla_g^2 u\big\|_{L^p(M)}+\|\nabla_g u\|_{L^p(M)}+\|u\|_{L^p(M)}\Big),\end{aligned}$$

since $|\Delta_g u|^2\leqslant n|\nabla_g^2 u|^2$, whence the equivalence follows. Consequently, $\|\cdot\|_{2,p}$ is a norm in $H^{2,p}(M)$ equivalent to $\|\cdot\|_{H^{2,p}(M)}$.

Now we are ready to prove the equivalence of the norms $\|\cdot\|_{H^{2,p}(M)}$ and

$$|||u|||_{H^{2,p}(M)}=\big(\|\Delta_g u\|^p_{L^p(M)}+\|u\|^p_{L^p(M)}\big)^{1/p}.$$

Proposition A1. *Let (M, g) be a smooth compact Riemannian manifold, with or without boundary. The norm $\|\cdot\|_{H^{2,p}(M)}$ is equivalent to the norm $|||\cdot|||_{H^{2,p}(M)}$.*

Proof. According to the above discussion, it suffices to show the equivalence of the norms $\|\cdot\|_{2,p}$ and $|||\cdot|||_{H^{2,p}(M)}$. For convenience of notation, denote $\eta_k u = (\eta_k u) \circ \phi_k$. By L^p-estimates for linear elliptic operators, for every $u \in H^{2,p}(M)$ there exists a constant $C > 0$ independent of u such that

$$\left\|\partial^2(\eta_k u)\right\|_{L^p(\Omega_k)} + \left\|\partial(\eta_k u)\right\|_{L^p(\Omega_k)} \leqslant C\left(\left\|\Delta_g(\eta_k u)\right\|_{L^p(\Omega_k)} + \|\eta_k u\|_{L^p(\Omega_k)}\right).$$

In the following, C will denote several possibly different constants independent of u. Expanding the right-hand side of this inequality, we get:

$$\begin{aligned}
&\left\|\partial^2(\eta_k u)\right\|_{L^p(\Omega_k)} + \left\|\partial(\eta_k u)\right\|_{L^p(\Omega_k)} \\
&\quad\leqslant C\left(\|\eta_k \Delta_g u\|_{L^p(\Omega_k)} + \left\|\langle\nabla_g \eta_k, \nabla_g u\rangle\right\|_{L^p(\Omega_k)} + \left\|(\Delta_g \eta_k)u\right\|_{L^p(\Omega_k)} + \|\eta_k u\|_{L^p(\Omega_k)}\right) \\
&\quad\leqslant C\left(\|\Delta_g u\|_{L^p(M)} + \|u\|_{L^p(M)}\right) + C\|\nabla_g u\|_{L^p(M)}.
\end{aligned} \tag{49}$$

On the other hand, by interpolation of lower derivatives norms in Sobolev spaces in open sets of the Euclidean space, we have:

$$\begin{aligned}
\|\nabla_g u\|^p_{L^p(M)} &= \left\|\left(\sum \eta_k\right)\nabla_g u\right\|^p_{L^p(M)} \leqslant N^p \sum \|\eta_k \nabla_g u\|^p_{L^p(M)} \\
&\leqslant C \sum \int_{\Omega_k} |\eta_k \nabla u|^p \,\mathrm{d}x \leqslant C \sum \left(\int_{\Omega_k} \left|\nabla(\eta_k u)\right|^p \mathrm{d}x + \int_{\Omega_k} \left|(\nabla \eta_k)u\right|^p \mathrm{d}x\right) \\
&\leqslant C \sum \left(\varepsilon \int_{\Omega_k} \left|\partial^2(\eta_k u)\right|^p \mathrm{d}x + C_\varepsilon \int_{\Omega_k} |\eta_k u|^p \,\mathrm{d}x\right) + C \int_M |u|^p \,\mathrm{d}v_g \\
&\leqslant C\varepsilon \sum \int_{\Omega_k} \left|\partial^2(\eta_k u)\right|^p \mathrm{d}x + C \int_M |u|^p \,\mathrm{d}v_g.
\end{aligned} \tag{50}$$

Hence, choosing ε small enough and combining (49) and (50), we obtain:

$$\begin{aligned}
\|u\|_{2,p} &\leqslant \sum\left(\left\|\partial^2(\eta_k u)\right\|_{L^p(\Omega_k)} + \left\|\partial(\eta_k u)\right\|_{L^p(\Omega_k)} + \|\eta_k u\|_{L^p(\Omega_k)}\right) \\
&\leqslant C\left(\|\Delta_g u\|_{L^p(M)} + \|u\|_{L^p(M)}\right).
\end{aligned}$$

The inequality in the opposite direction follows immediately from a simple computation. □

Proposition A2. *Let (M, g) be a compact Riemannian manifold with boundary. In the Sobolev spaces $H_0^{2,p}(M)$ and $H^{2,p}(M) \cap H_0^{1,p}(M)$, the norm $|||\cdot|||_{H^{2,p}(M)}$ is equivalent to the norm*

$$||||u||||_{H^{2,p}(M)} = \|\Delta_g u\|_{L^p(M)}.$$

Proof. Using Theorem 3.72 of [3] and mimicking the proof of Lemma 9.17 of [18], we obtain that there exists a constant C independent of u such that

$$|||u|||_{H^{2,p}(M)} \leqslant C\|\Delta_g u\|_{L^p(M)}. \qquad \square$$

Appendix B. Asymptotic behaviors

The asymptotic behavior of the minimizers of the Sobolev quotient (4) and their Laplacians was established by Hulshof and van der Vorst in [23]. Their estimates there were written in a form appropriate to the problem they were dealing with, namely, elliptic systems. We rewrite them here in the form most suitable to our needs. Given a positive radial extremal function $z(r)$ for (4), there exist positive constants $C_j = C_j(n, p)$ such that

$$\begin{cases} \lim\limits_{r\to\infty} r^{n-2}z(r) = C_1 & \text{if } 1 < p < 2\dfrac{n-1}{n}, \\ \lim\limits_{r\to\infty} \dfrac{r^{n-2}}{\ln r}z(r) = C_2 & \text{if } p = 2\dfrac{n-1}{n}, \\ \lim\limits_{r\to\infty} r^{(n-2p)/(p-1)}z(r) = C_3 & \text{if } 2\dfrac{n-1}{n} < p < \dfrac{n}{2}, \end{cases} \tag{51}$$

$$\begin{cases} \lim\limits_{r\to\infty} r^{n^2/(n-2p)}\big(-\Delta z(r)\big) = C_4 & \text{if } 1 < p < \dfrac{2n(n-1)}{n^2+2n-4}, \\ \lim\limits_{r\to\infty} \dfrac{r^{(n-2)/(p-1)}}{(\ln r)^{1/(p-1)}}\big(-\Delta z(r)\big) = C_5 & \text{if } p = \dfrac{2n(n-1)}{n^2+2n-4}, \\ \lim\limits_{r\to\infty} r^{(n-2)/(p-1)}\big(-\Delta z(r)\big) = C_6 & \text{if } \dfrac{2n(n-1)}{n^2+2n-4} < p < \dfrac{n}{2}, \end{cases} \tag{52}$$

and

$$\begin{cases} \lim\limits_{r\to\infty} r^{n-1}\big(-z'(r)\big) = C_7 & \text{if } 1 < p < 2\dfrac{n-1}{n}, \\ \lim\limits_{r\to\infty} \dfrac{r^{n-1}}{\ln r}\big(-z'(r)\big) = C_8 & \text{if } p = 2\dfrac{n-1}{n}, \\ \lim\limits_{r\to\infty} r^{(n-p-1)/(p-1)}\big(-z'(r)\big) = C_9 & \text{if } 2\dfrac{n-1}{n} < p < \dfrac{n}{2}. \end{cases} \tag{53}$$

The decay (53) follows from (52) using

$$z'(r) = \frac{1}{r^{n-1}} \int_0^r \Delta z(s) s^{n-1} \, \mathrm{d}s.$$

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